② $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad x \in (0,1),\ t \geq 0,\ a > 0$

Initial conditions:

$$u(x,0) = \sin(2\pi x)$$

Periodic boundary conditions:

$$u(0,t) = u(1,t)$$

a) Implicit finite difference scheme.

$$\left.\frac{\partial u}{\partial t}\right|_i^{n+1} = \frac{U_i^{n+1} - U_i^n}{\Delta t} + O(\Delta t) \quad \text{(Backward approximation)}$$

$$\left.\frac{\partial u}{\partial x}\right|_i^{n+1} = \frac{U_{i+1}^{n+1} - U_i^{n+1}}{\Delta x} + O(\Delta x) \quad \text{(Forward approximation)}$$

<u>FDE</u>

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + a\left(\frac{U_{i+1}^{n+1} - U_i^{n+1}}{\Delta x}\right) + O(\Delta t, \Delta x) = 0$$

$$U_i^{n+1} - U_i^n + ar\left(U_{i+1}^{n+1} - U_i^{n+1}\right) = U_i^n$$

$$\boxed{(1-ar)\,U_i^{n+1} + (ar)\,U_{i+1}^{n+1} = U_i^n}$$

Resulting equation system: $\underline{K}\,\underline{U}^{n+1} = \underline{U}^{n}$

$$\underline{K} = \begin{bmatrix} (1-ar) & ar & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & (1-ar) & ar & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & (1-ar) & ar & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & & (1-ar) & ar & 0 \\ 0 & 0 & 0 & \cdots & & 0 & (1-ar) & ar \\ -1 & 0 & 0 & \cdots & & 0 & 0 & -1 \end{bmatrix}$$

$$\underline{U}^{n+1} = \begin{Bmatrix} U_0^{n+1} \\ U_1^{n+1} \\ \vdots \\ U_M^{n+1} \\ U_{M+1}^{n+1} \end{Bmatrix} \qquad \underline{U}^{n} = \begin{Bmatrix} U_0^{n} \\ U_1^{n} \\ \vdots \\ U_M^{n} \\ U_M^{n+1} \end{Bmatrix}$$

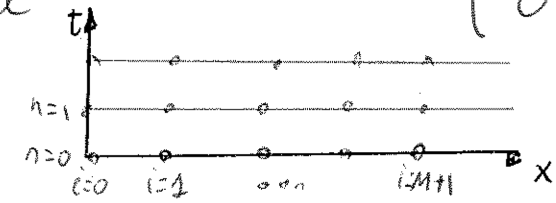
Suggest a direct method and an iterative method.

Direct method $\Rightarrow$ Cholesky Factorization

Iterative method $\Rightarrow$ Jacobi method.

④ $\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} - \sigma u = 0$ in $x \in (0,1)$, $t > 0$

$$BC \begin{cases} u(0,t) = 0 \\ \frac{\partial u}{\partial x}(1,t) = 0 \end{cases} \qquad IC \begin{cases} 0 & x < 1/4 \\ 4x-1 & 1/4 \le x < 1/2 \\ -4x+3 & 1/2 \le x < 3/4 \\ 0 & 3/4 \le x \end{cases}$$



a) Propose an explicit finite difference scheme for the solution of PDE. Detail the numerical treatment of boundary conditions.

FDA

$$\left.\frac{\partial u}{\partial t}\right|_i^n = \frac{u_i^{n+1} - U_i^n}{\Delta t} \quad \text{(Forward in time)}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_i^n = \frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{\Delta x^2} \quad \text{(Centered in space)}$$

FDE

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} - \nu \frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{\Delta x^2} - \sigma U_i^n = 0$$

$$U_i^{n+1} = \left[\sigma U_i^n + \nu \frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{\Delta x^2}\right] \Delta t + U_i^n \qquad r = \frac{\Delta t}{\Delta x^2}$$

$$\boxed{U_i^{n+1} = (\nu r) U_{i-1}^n + (\sigma - 2\nu r + 1) U_i^n + (\nu r) U_{i+1}^n}$$

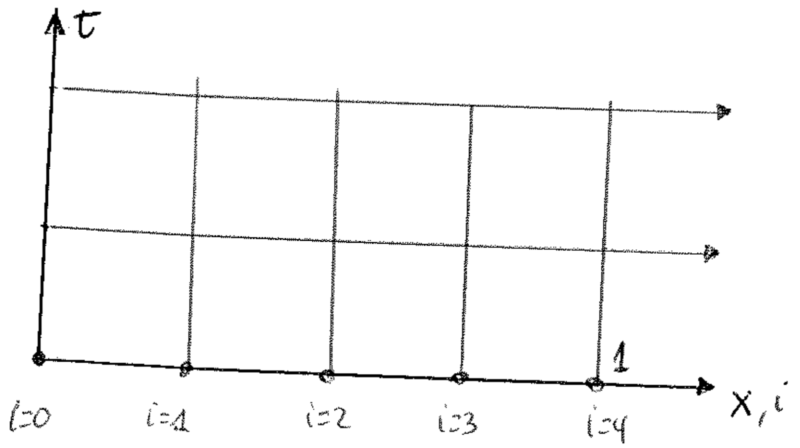
b) Scheme for $\sigma=0$

$$U_i^{n+1} = (\nu r)\, U_{i-1}^n + (1-2\nu r)\, U_i^n + (\nu r)\, U_{i+1}^n$$

Scheme for $\nu=0$

$$U_i^{n+1} = (\sigma+1)\, U_i^n$$

c)



$\Delta\tau = 0'1$

$\nu = 0'1$

$\sigma = -0'1$

$\Delta x = 0'25$

$i = M+1$

$$(-\upsilon r)\, U_M^{n+1} + (1-2\upsilon r-\sigma\Delta t)\, U_{M+1}^{n+1} - (\upsilon r)\, U_{M+2}^{n+1} = U_{M+1}^{n}$$

$$(-\upsilon r)\, U_M^{n+1} + (1-2\upsilon r-\sigma\Delta t)\, U_{M+1}^{n+1} - (\upsilon r)\left[\Delta x \cdot U_x(1,t) + U_{M+1}^{n}\right] = U_{M+1}^{n}$$

$$\boxed{(-\upsilon r)\, U_M^{n+1} + (1-3\upsilon r-\sigma\Delta t)\, U_{M+1}^{n+1} - \upsilon r\,\Delta x \cdot U_x(1,t) = U_{M+1}^{n}}$$

So, the system of equations takes the following form: $\underline{\underline{K}}\,\underline{U}^{n+1} = \underline{F}$

$$\underline{\underline{K}} \begin{bmatrix} (1-2\upsilon r-\sigma\Delta t) & -\upsilon r & 0 & \cdots & 0 & 0 & 0 \\ -\upsilon r & (1-2\upsilon r-\sigma\Delta t) & -\upsilon r & \cdots & 0 & 0 & 0 \\ 0 & -\upsilon r & (1-2\upsilon r-\sigma\Delta t) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\upsilon r & (1-2\upsilon r-\sigma\Delta t) & -\upsilon r \\ 0 & 0 & 0 & \cdots & 0 & -\upsilon r & (1-3\upsilon r-\sigma\Delta t) \end{bmatrix}$$

$$\underline{U}^{n+1} = \begin{Bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{M+1}^{n+1} \end{Bmatrix} \qquad \underline{F} = \begin{Bmatrix} U_1^{n} + \upsilon r\, U_0^{n+1} \\ U_2^{n} \\ \vdots \\ U_{M+1}^{n} + \upsilon r\,\Delta x \cdot U_x(1,t) \end{Bmatrix}$$