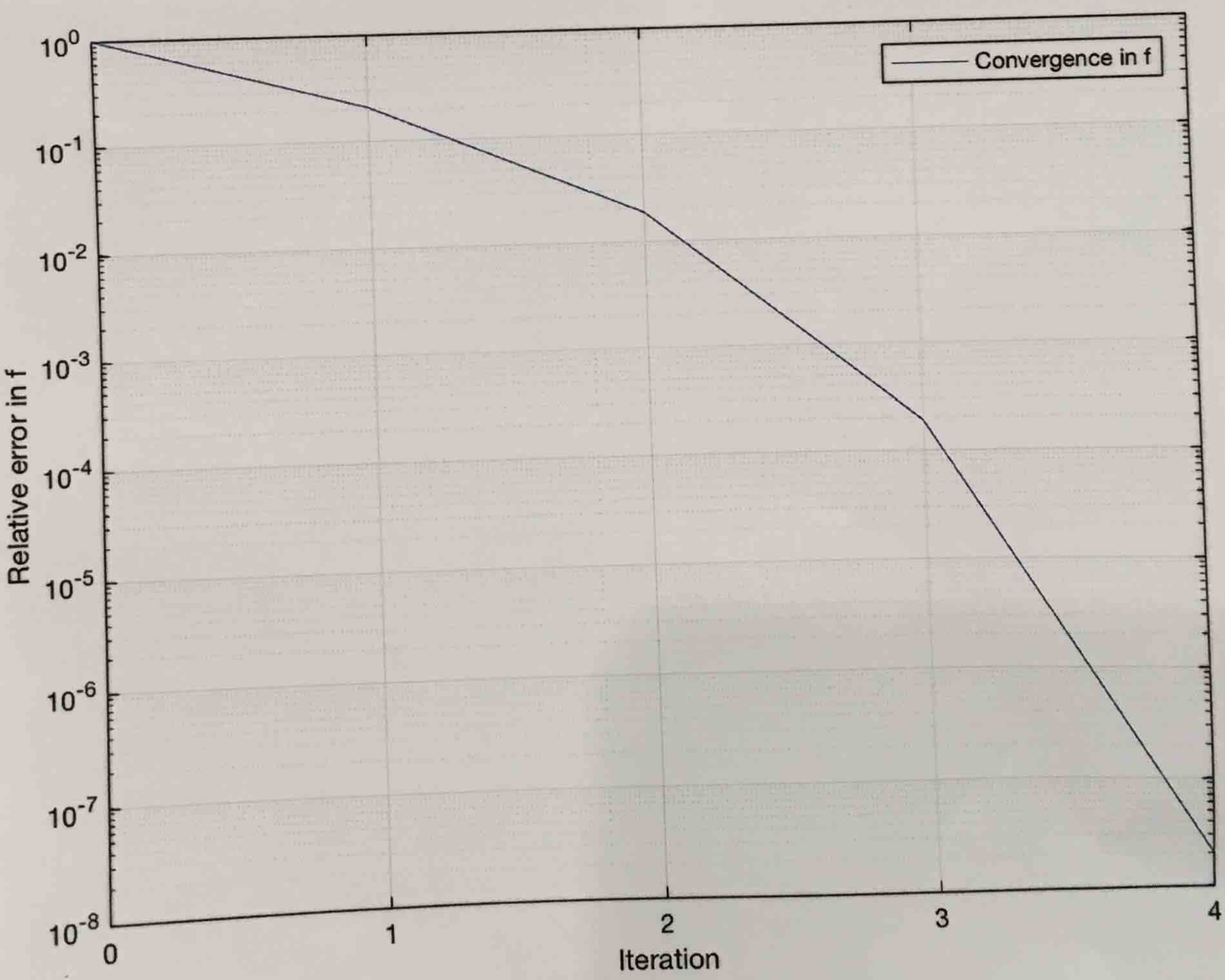
NUMERICAL METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS:

1 Newton Method: $f(x) := x^3 + 2x^2 + 10x - 20 = 0$

| $x_i$ | $f(x_i)$ | $f'(x_i)$ | $\Delta x_{i+1} = -\frac{f(x_i)}{f'(x_i)}$ |
|---|---|---|---|
| $\sqrt[3]{20}$ | 41,8803 | 42,9619 | -0,9748 |
| 1,7396 | 8,7126 | 26,0369 | -0,3346 |
| 1,4050 | 0,7708 | 21,5417 | -0,0358 |
| 1,3692 | 0,0079 | 21,1007 | -0,000375 |
| 1,3688 | $8{,}59 \cdot 10^{-7}$ | | |

Convergence plot: As it can be seen in the convergence plot, the method behaves as expected.



(5) $3^{rd}$ ORDER QUADRATURES IN $(0,1)$ :

(a) minimum NUMBER OF INTEGRATION POINTS. POINTS & WEIGHTS :

$3^{rd}$ order — $3+1$ dof — $\boxed{\begin{matrix}\text{2 POINTS}\\ \text{2 WEIGHTS}\end{matrix}}$

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\int_0^1 p(x)dx = a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} = w_0 \cdot p(x_0) + w_1 \cdot p(x_1) =$$

$$= w_0 (a_0 + a_1 x_0 + a_2 x_0^2 + a_3 x_0^3) + w_1 \cdot (a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3) =$$

$$= a_0 (w_0 + w_1) + a_1 \cdot (w_0 x_0 + w_1 x_1) + a_2 (w_0 x_0^2 + w_1 x_1^2) + a_3 (w_0 x_0^3 + w_1 x_1^3)$$

$$\Downarrow$$

$$\left.\begin{aligned} w_0 + w_1 &= 1 \\ w_0 x_0 + w_1 x_1 &= \frac{1}{2} \\ w_0 x_0^2 + w_1 x_1^2 &= \frac{1}{3} \\ w_0 x_0^3 + w_1 x_1^3 &= \frac{1}{4} \end{aligned}\right] \text{— NON LINEAR SYSTEM OF EQUATIONS.}$$

$$\boxed{\begin{matrix} x_0 = 0{,}2113 & x_1 = 0{,}7887 \\ w_0 = 0{,}500 & w_1 = 0{,}500 \end{matrix}}$$

MATLAB fsolve

(b) $x_0 = \frac{1}{4}$

$x_1 = \frac{1}{2}$

$x_2 = \frac{3}{4}$

$x_3 = 1$

dot → $w_0, w_1, w_2, w_3$ → max order 3

↓

It is possible.

$$\int_0^1 (a_0 + a_1 x + a_2 x^2 + a_3 x^3)\, dx = a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} =$$

$$= a_0 \cdot (w_0 + w_1 + w_2 + w_3) + a_1 (w_0 \cdot \tfrac{1}{4} + w_1 \cdot \tfrac{1}{2} + w_2 \cdot \tfrac{3}{4} + w_3 \cdot 1) +$$

$$+ a_2 (w_0 \cdot \tfrac{1}{16} + w_1 \cdot \tfrac{1}{4} + w_2 \cdot \tfrac{9}{16} + w_3 \cdot 1) +$$

$$+ a_3 \cdot (w_0 \cdot \tfrac{1}{64} + w_1 \cdot \tfrac{1}{8} + w_2 \cdot \tfrac{27}{64} + w_3 \cdot 1)$$

⇩

$$1 = w_1 + w_2 + w_3 + w_4$$

$$\frac{1}{2} = \frac{1}{4} w_0 + \frac{1}{2} w_1 + \frac{3}{4} w_2 + w_3$$

$$\frac{1}{3} = \frac{1}{16} w_0 + \frac{1}{4} w_1 + \frac{9}{16} w_2 + w_3$$

$$\frac{1}{4} = \frac{1}{64} w_0 + \frac{1}{8} w_1 + \frac{27}{64} w_2 + w_3.$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1/4 & 1/2 & 3/4 & 1 \\ 1/16 & 1/4 & 9/16 & 1 \\ 1/64 & 1/8 & 27/64 & 1 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \end{pmatrix}$$

⇩

$$\boxed{\begin{aligned} w_0 &= 2/3 \\ w_1 &= -1/3 \\ w_2 &= 2/3 \\ w_3 &= 0 \end{aligned}}$$

$$\Rightarrow \boxed{I \doteq \frac{2}{3} f\left(\frac{1}{4}\right) - \frac{1}{3} f\left(\frac{1}{2}\right) + \frac{2}{3} f\left(\frac{3}{4}\right)}$$

(6) (a) n+1 Gaussian quadrature points:

$(n+1)\cdot 2$ dof $= 2n+2$ → Polinomials of order $\underline{\underline{2n-1}}$

(b) $2\cdot 2-1 = 3^{rd}$ order polinomials max:

i) $\int_0^1 \sin x \, dx$ → NOT EXACT  ($\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \dots$)

ii) $\int_0^1 x^3 \, dx$ → EXACT ✓

iii) $\int_0^1 x^4 \, dx$ → NOT EXACT

iv) $\int_0^1 x^{5,5} \, dx$ → NOT EXACT.

(7) (a) trapezoidal rule over 2 uniform intervals:

$I \approx \frac{1}{4} \cdot (f(0) + 2f(\frac{1}{2}) + f(1))$

$\int_0^1 12x \, dx = \frac{1}{4} \cdot (0 + 2\cdot 6 + 12) = 6$ → EXACT

$\int_0^1 (5x^3 + 2x) dx = \frac{1}{4} \cdot (0 + 2\cdot\frac{5}{8} + 2\cdot 2 + 7) = \frac{49}{16} = 3{,}0625$ → NOT EXACT

(b) Simpsons rule over 2 uniform intervals:

$I \approx \frac{1}{12} ( f(0) + 4f(0{,}25) + 2f(0{,}5) + 4f(0{,}75) + f(1))$

$\int_0^1 12x \, dx = \frac{1}{12} \cdot (0 + 4\cdot 3 + 2\cdot 6 + 4\cdot 9 + 12) = 6$ — EXACT

$\int_0^1 (5x^3 + 2x) \, dx = \frac{1}{2} \left(0 + \frac{4\cdot 37}{64} + \frac{2\cdot 13}{8} + \frac{4\cdot 231}{64} + 7\right) = \frac{9}{4}$ — EXACT.

- Exacts: $\int_0^1 12x \, dx = 6$ | Simpson's order 3 ; trapez order 1 → OK, As expected.

$\int_0^1 (5x^3 + 2x) \, dx = 9/4$

(10) $\int_0^1 \int_0^1 (9x^3 - 8x^2)(y^3 + y) \, dx \, dy$ ; Simpson's, $I \approx \frac{h}{3} (f(x_0) + 4f(x_1) + f(x_2))$

$I \approx \int_0^1 \frac{1}{6} [ f(0,y) + 4f(\tfrac{1}{2},y) + f(1,y)] \, dy =$

$= \frac{1}{36} \Big[f(0,0) + 4f(\tfrac{1}{2},0) + f(1,0) + 4f(0,\tfrac{1}{2}) + 16f(\tfrac{1}{2},\tfrac{1}{2}) + 4f(1,\tfrac{1}{2}) + f(0,1) + 4f(\tfrac{1}{2},1) + f(1,1)\Big] =$

$= \frac{1}{36} \cdot (16\cdot 1{,}9531 + 4\cdot 10{,}6250 + 4\cdot 6{,}25 + 34) = 3{,}6875.$

$I = \int_0^1 (y^3+y) \left( \frac{9x^4}{4} - \frac{8x^3}{3}\right)\Big|_{x=0}^{x=1} dy = \left(\frac{9}{4} - \frac{8}{3}\right) \cdot \left[\frac{y^4}{4} + \frac{y^2}{2}\right]_0^1 = \left(\frac{9}{4} - \frac{8}{3}\right)\left(\frac{1}{4}+\frac{1}{2}\right) = 3{,}6875$

Simpson's → order 3 → EXACT AS EXPECTED