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1.

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

$L = 1m$

$g = 9{,}8\, m/s^2$

$\theta(1) = 0{,}4$ rad/s

$\frac{d\theta}{dt}(1) = 0$ rad/s

a) $\theta(0)$?

$h = -0{,}5$ and $h = -0{,}25$

Reducing the order of the ODE: $\begin{cases} \frac{d\theta}{dt} = \omega \\ \frac{d\omega}{dt} = -\frac{g}{L}\theta \end{cases}$

Applying Heun's method (RK2):

$$\left.\begin{aligned} \theta_{i+1} &= \theta_i + (k_\theta^1 + k_\theta^2)\cdot\frac{h}{2} \\ \omega_{i+1} &= \omega_i + (k_\omega^1 + k_\omega^2)\cdot\frac{h}{2} \end{aligned}\right\} \text{where} \quad \begin{aligned} k_\theta^1 &= \omega_i \\ k_\theta^2 &= \omega_i - \frac{g}{L}\theta_i\cdot h \\ k_\omega^1 &= -\frac{g}{L}\theta_i \\ k_\omega^2 &= -\frac{g}{L}(\theta_i + \omega_i h) \end{aligned}$$

For $h = -0{,}5$

$\theta_0 = 0{,}4$ rad/s

$\theta_1 = -0{,}09$ rad/s

$\theta_2 = \boxed{-0{,}9598 \text{ rad/s} = \theta(0)}$

For $h = -0{,}25$

$\theta_0 = 0{,}4$ rad/s

$\theta_1 = 0{,}2775$ rad/s

$\theta_2 = -0{,}05248$ rad/s

$\theta_3 = -0{,}37635$ rad/s

$\theta_4 = \boxed{-0{,}46478 \text{ rad/s} = \theta(0)}$

b) Aproximation of ε for $h = -0{,}5$

The relative error ε can be ~~computed~~ approximated using the results given by $h = -0{,}5$ and $h = -0{,}25$, where the latter will be a better approximation of $\theta(0)$.

$$\varepsilon = \left|\frac{\theta(0)|_{h=-0{,}5} - \theta(0)|_{h=-0{,}25}}{\theta(0)|_{h=-0{,}25}}\right| = 1{,}06$$

c) h to obtain $\varepsilon = 10^{-3}$

$$h^* = \left(\frac{10^{-3}}{\varepsilon}\right)^{\frac{1}{p+1}}\cdot h \Rightarrow \text{Since Heun is a 2nd order method} \Rightarrow h^* = \left(\frac{10^{-3}}{1{,}06}\right)^2\cdot 0{,}5 =$$

2.

$$\frac{dy}{dx} = y - x^2 + 1 \quad x \in (0,1)$$

$y(0) = 1$

a) Euler with $h = 0{,}25$

$y_{i+1} = y_i + \underbrace{f(x_i, y_i)}_{= y_i - x_i^2 + 1} \cdot h$

$y_0 = 1$
$y_1 = 1{,}5$
$y_2 = 2{,}1094$
$y_3 = 2{,}8242$
$\boxed{y_4 = 3{,}6396}$

b) Heun with a step such that the computational cost is equivalent to a)

In Heun's method, for each ~~ea~~ iteration the function has to be ~~computed~~ evaluated twice; whereas in Euler's method the function is evaluated once $\Rightarrow$ In Heun's method $h = 0{,}5$ in order to get the same computational cost that in Euler's.

- $y_0 = 1$; $k_1 = f(y_0, x_0) = 2$; $k_2 = f(y_0 + h k_1, x_0 + h) = 2{,}75$
- $y_1 = 2{,}1875$; $k_1 = 2{,}9375$; $k_2 = 3{,}65625$
- $\boxed{y_2 = 3{,}8359}$

c) Interpolation polynomial for b)

$$\left.\begin{aligned} A x_0^2 + B x_0 + C &= y_0 \\ A x_1^2 + B x_1 + C &= y_1 \\ A x_2^2 + B x_2 + C &= y_2 \end{aligned}\right\} \quad \begin{aligned} C &= 1 \\ 0{,}25A + 0{,}5B + 1 &= 2{,}1875 \\ A + B + 1 &= 3{,}8359 \end{aligned} \quad \left\{ \begin{aligned} A &= 1{,}5833 \\ B &= 1{,}2526 \\ C &= 1 \end{aligned}\right.$$

d) Interpolation polynomial for a)

Since the ~~exact~~ analytical solution is a 2nd degree polynomial, only 3 points are needed in order to do the interpolation. Using the initial, middle and last point obtained in a):

$$\left.\begin{aligned} A x_0^2 + B x_0 + C &= y_0 \\ A x_2^2 + B x_2 + C &= y_2 \\ A x_4^2 + B x_4 + C &= y_4 \end{aligned}\right\} \quad \begin{aligned} C &= 1 \\ 0{,}25A + 0{,}5B + 1 &= 2{,}1094 \\ A + B + 1 &= 3{,}6396 \end{aligned} \quad \left\{ \begin{aligned} A &= 1{,}4792 \\ B &= 1{,}1604 \\ C &= 1 \end{aligned}\right.$$

3.

a) Truncation error for the Forward Euler method

$$y_{i+1} \overset{\text{Taylor}}{\approx} y_i + h\frac{dy}{dx}(x_i) + O(h^2) \; , \quad \frac{dy}{dx}(x_i) = \frac{y_{i+1} - y_i - O(h^2)}{h} = \frac{y_{i+1}-y_i}{h} + \underbrace{O(h)}_{\text{Truncation error}}$$

b) Backward Euler method

For $\frac{dy}{dx} = f(x,y)$, the Backward Euler method can be written as:

$y_{i+1} = y_i + h f(x_{i+1}, y_{i+1})$, which is an equation to be solved for y_{i+1}

c) Stability limits for the Forward and Backward Euler methods for $\frac{dy}{dx} = -\lambda y \quad (\lambda > 0)$

Forward

$y_{i+1} = y_i + f(x_i, y_i)h = y_i - \lambda h y_i = y_i(1 - \lambda h)$

The solution is stable if $|1 - \lambda h| < 1$ because the analytical solution $y = y_0 e^{-\lambda x}$ goes to zero for $x \to \infty$. For $\lambda \in \mathbb{R}$ $-1 < 1 - \lambda h < 1$

Backward

$y_{i+1} = y_i - \lambda h y_{i+1} \; , \quad y_{i+1} = \frac{1}{1+\lambda h} y_i$

The solution is stable if $\frac{1}{1+\lambda h} < 1 \Rightarrow |1 + \lambda h| > 1$

d) Backward Euler to solve $\frac{dy}{dx} = -25y^{3,5}$, $y(0)$; $h = \frac{1}{10}$ in 2 steps

- $y_0 = 1$
- $y_1 = 1 - 2,5y_1^{3,5}$; $2,5y_1^{3,5} + y_1 - 1 = 0$. Applying Newtons method for an initial guess $y_1 = 0,6$:

$$y_1^1 = 1 - \frac{2,5y_1^{3,5} + y_1 + 1}{8,75y_1^{2,5} + 1} = 0,59469$$

$$y_1^2 = 1 - \frac{2,5(y_1^1)^{3,5} + y_1^1 + 1}{8,75(y_1^1)^{2,5} + 1} = 0,59464 \equiv y_1$$

- $y_2 = y_1 - 2,5y_2^{3,5}$; $2,5y_2^{3,5} + y_2 - y_1 = 0$. Applying Newtons method for an initial guess $y_2 = 0,45$:

$y_2^1 = 0,44626$

$y_2^2 = \boxed{0,44624 \equiv y_2}$

e) Forward Euler method to solve $\frac{dy}{dx} = -25y^{3,5}$, $y(0)$, $h = \frac{1}{10}$

The solution won't be stable

Since $|1 - h\lambda| = |1 - 2,5| = 1,5 > 1 \Rightarrow$ The solution won't be stable