2016.11.20

PDE. Basic.

Q1. Find the Root of. $f(x) := x^3+2x^2+10x-20=0$ using Newton's Method with 4 iteration.

Solution:

Function. $f(x) := x^3+2x^2+10x-20$.

Derivative. $f'(x) = 3x^2+4x+10$

The initial guess given by the question is. $x^{(0)} = \sqrt[3]{20}$

Iter 1.

$$\Delta x^{(1)} = -\frac{f(x_0)}{f'(x_0)} = -\frac{(\sqrt[3]{20})^3+2(\sqrt[3]{20})^2+10(\sqrt[3]{20})-20}{3\times(\sqrt[3]{20})^2+4\times\sqrt[3]{20}+10} = -0.9748.$$

$$x^{(1)} = x^{(0)}+\Delta x^{(1)} = \sqrt[3]{20}+(-0.9748) = 1.7396.$$

Iter 2.

$$\Delta x^{(2)} = -\frac{f(x^{(1)})}{f'(x^{(1)})} = -\frac{(1.7396)^3+2\times(1.7396)^2+10\times1.7396-20}{3\times(1.7396)^2+4\times1.7396+10} = -0.3346$$

$$x^{(2)} = x^{(1)}+\Delta x^{(2)} = 1.7396-0.3346 = 1.4050.$$

Iter 3.

$$\Delta x^{(3)} = -\frac{f(x^{(2)})}{f'(x^{(2)})} = -\frac{(1.4050)^3+2\times(1.4050)^2+10\times1.4050-20}{3\times(1.4050)^2+4\times1.4050+10} = -0.0358.$$

$$x^{(3)} = x^{(2)}+\Delta x^{(3)} = -0.0358+1.4050 = 1.3692.$$

Iter 4.

$$\Delta x^{(4)} = -\frac{f(x^{(3)})}{f'(x^{(3)})} = -\frac{(1.3692)^3+2\times(1.3692)^2+10\times(1.3692)-20}{3\times(1.3692)^2+4\times1.3692+10} = -3.7498\times10^{-4}$$

$$x^{(4)} = x^{(3)}+\Delta x^{(4)} = 1.3692-3.7498\times10^{-4} = 1.3688$$

*. Find the exact root of the equation.

$f(x) = x^3+2x^2+10x-20$,

coefficient. $a=1$. $b=2$. $c=10$. $d=-20$.

The discriminant.

$$\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 = -20960.$$

$\Delta < 0$, the equation has one real root and 2 none real complex conjugate root.

$$\Delta_0 = b^2 - 3ac = -26.$$

$$\Delta_1 = 2b^3 - 9abc + 27a^2d = -704.$$

$$C = \sqrt[3]{\frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}} = 2.89 \text{ and } 4.4982 + 7.7924i$$

Ignore the real complex number.

the root. $x = -\frac{1}{3a}\left(b + C + \frac{\Delta_0}{C}\right) = 1.3688.$

Relative Error.

Inorder to obtain the convergence graph, which is relative error verses iteration.

The relative error is obtained as.

$$\text{relative Error} = \frac{|\text{Absolute result} - \text{Approximate result}|}{\text{Absolute result}}.$$

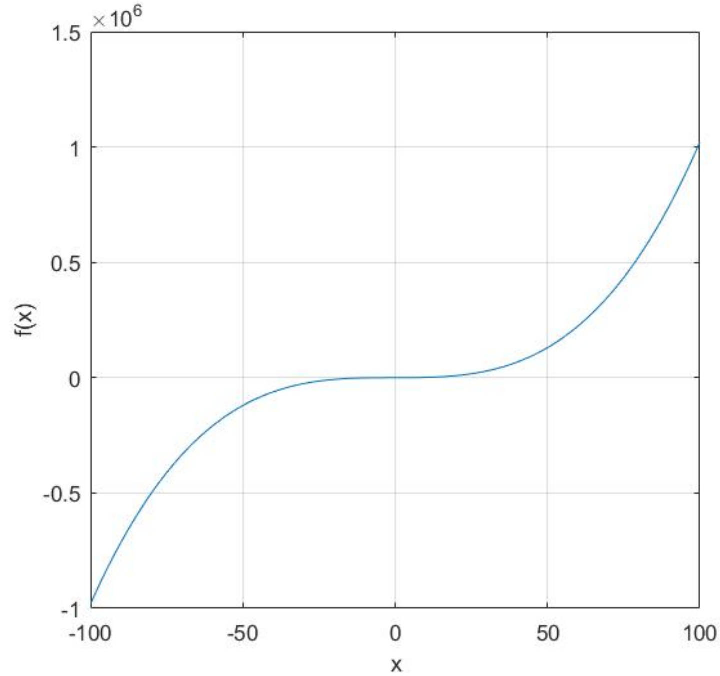


Figure 1 Function f(x)

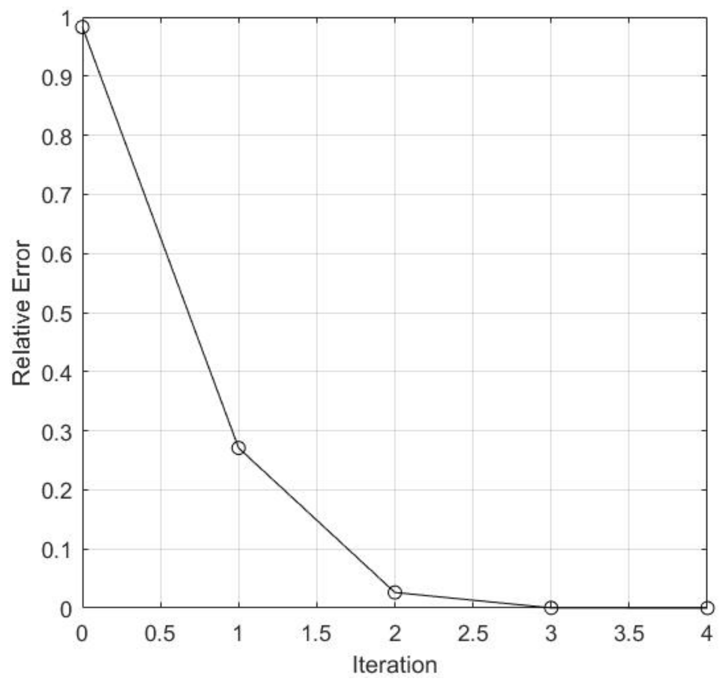


Figure 2 The convergence graph

The convergence graph of the Newton-Raphson Method applied for the function $f(x)$ is presented in Figure 2. As shown in Figure 2, the convergence of Newton-Raphson Method is fast, the scale of the error decrease quadratically during each iteration. Newton-Raphson Method behaves as expected.

Q5. Third order quadratures in (0, 1).

(a). The minimum number of integration points.

According to Gauss quadratures,

The degree of polynomal that the Gauss quadrature can integrate exactly is $2n+1$

So, for third order quadrature.

$2n+1 = 3. \quad \Rightarrow n = 1$

the integration points and weights.

integration points	$+\sqrt{1/3}$.	$-\sqrt{1/3}$.
weights.	1.	1.

(b).

Yes, it is possible.

~~$P_0(x)=1$. $\int_0^1 x\,dx = 1$~~ $\int_0^1 P(x)dx = \sum_{i=0}^{3} w_i P(x_i)$

$P_0(x) = 1.$, $\int_0^1 1\,dx = 1 = w_0 P(x_0) + w_1 P(x_1) + w_2 P(x_2) + w_3 P(x_3)$

$= w_0 + w_1 + w_2 + w_3.$

$P_1(x) = x$, $\int_0^1 x\,dx = \frac{1}{2} = w_0 \times \frac{1}{4} + w_1 \times \frac{1}{2} + w_2 \times \frac{3}{4} + w_3.$

$P_2(x) = x^2$, $\int_0^1 x^2 dx = \frac{1}{3} = w_0 \times \frac{1}{16} + w_1 \times \frac{1}{4} + w_2 \times \frac{9}{16} + w_3.$

$P_3(x) = x^3$, $\int_0^1 x^3 dx = \frac{1}{4} = w_0 \times \frac{1}{64} + w_1 \times \frac{1}{8} + w_2 \times \frac{27}{64} + w_3.$

Solve the 4 equations above and obtain the weights.

$w_0 = 2/3$. $w_1 = -1/3$. $w_2 = 2/3$. $w_4 = 0$.

Q6.

(a). points $= n+1$.

Gaussian quadrature that is integrated exactly.

$2(n+1)+1 = 2n+3$.

(b). $n=2$, $2n+1 = 5$.

i). $\int_0^1 \sin x \, dx$. – cannot be integrate exactly.

ii) $\int_0^1 x^3 dx$. – can be integrate exactly

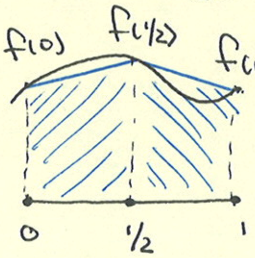
iii) $\int_0^1 x^4 dx$. – can be integrate exactly

iv). $\int_0^1 x^{5.5} dx$ – cannot be integrate exactly

Q7. Integrate over 2 uniform intervals

i). Trapezoidal.

$\rightarrow \int_0^1 12x \, dx = [f(0)+f(1/2)] \times \frac{1/2}{2} + [f(1/2)+f(1)] \times \frac{1/2}{2}$



$\Rightarrow f(0) = 0$. $f(1/2) = 6$. $f(1) = 12$.

Atrap. $= (0+6)\times\frac{1}{4} + (6+12)\times\frac{1}{4} = 6$.

$\rightarrow \int_0^1 (5x^3+2x)dx = [f(0)+f(1/2)]\times\frac{1/2}{2} + [f(1/2)+f(1)]\times\frac{1/2}{2}$

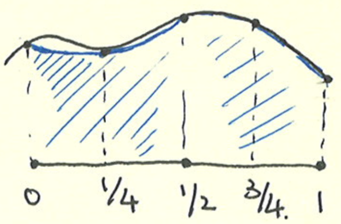
$\Rightarrow f(0) = 0$. $f(1/2) = 1.625$. $f(1) = 7$.

Atrap. $= (0+1.625)\times\frac{1}{4} + (1.625+7)\times\frac{1}{4} = 41/16$.

ii). Simpson's.

$\rightarrow \int_0^1 12x \, dx = [f(0)+4\times f(1/4)+f(1/2)]\times\frac{1/2}{6} +$

$[f(1/2)+4\times f(3/4)+f(1)]\times\frac{1/2}{6}$



$f(0) = 0$. $f(1/4) = 3$. $f(1/2) = 6$. $f(3/4) = 9$. $f(1) = 12$

Asimp. $= (0+4\times3+6)\times\frac{1}{12} + (6+9\times4+12)\times\frac{1}{12} = 6$.

$\longrightarrow \int_0^1 5x^3+2x\,dx = [f(0)+4\times f(1/4)+f(1/2)]\times\frac{1/2}{6} + [f(1/2)+4\times f(3/4)+f(1)]\times\frac{1/2}{6}$

$f(0)=0.\ f(1/4)=37/64.\ f(2/4)=13/8.\ f(3/4)=231/64.\ f(1)=7.$

$A_{simp.} = [0+4\times 37/64 + 13/8]\times\frac{1}{12} + [13/8 + 231/64\times 4 + 7]\times\frac{1}{12} = 9/4.$

(3) Compare the Error.

Exact Solution : $\int_0^1 12x\,dx = [6x^2]_0^1 = 6.$

$\int_0^1 5x^3+2x\,dx = [\frac{5}{4}x^4+x^2]_0^1 = \frac{9}{4}.$

Relative Error of Trapezoidal

For. $\int_0^1 12x\,dx$, $E_{trap,1} = \frac{|6-6|}{6} = 0.$

For $\int_0^1 5x^3+x^2dx$. $E_{trap,2} = \frac{|9/4 - 41/16|}{9/4} = \frac{5}{36}$

Relative Error of Simpson's

For $\int_0^1 12x\,dx$ $E_{simp.1} = \frac{|6-6|}{6} = 0.$

For $\int_0^1 5x^3+x^2dx$ $E_{simp,2} = \frac{|9/4 - 9/4|}{9/4} = 0$

* Methods ~~behaving~~ as expected.

Trapezoidal is capable of integrate the linear equation but not the polynomial ~~up to~~ higher than quadratic equation. And Simpson can integrate ~~with~~ accurately the polynomial ~~high~~ with order 3. (cubic equation).

Q10. Using Simpson's rule to integrate

$$\int_0^1\int_0^1 (9x^3+8x^2)(y^3+y)\,dx\,dy.$$

Solution

Simpson's Rule for double integrals.

$$\int_a^b\int_c^d f(x,y)\,dx\,dy.$$

is given by.

$$S_{mn} = \frac{(b-a)(d-c)}{9mn} \sum_{i,j=0,0}^{m,n} W_{i+1,j+1} f(x_i,y_j).$$

where the weight.

$$W = \begin{bmatrix} 1 & 4 & 2 & 4 & \cdots & 4 & 1 \\ 4 & 16 & 8 & 16 & \cdots & 16 & 4 \\ 2 & 8 & 4 & 8 & \cdots & 8 & 2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 4 & 2 & 4 & \cdots & 4 & 1 \end{bmatrix}$$

and m, n are the subdivisions of the domains along x and y.

For our problem, $m=n=2$, $a=c=0$, $b=d=1$.

One interval with 3 point on both x and y direction.

$x_1=0$. $x_2=1/2$. $x_3=1$. $y_1=0$. $y_2=1/2$. $y_3=1$.

$f(x_1,y_1)=f(0,0)=0$. $f(x_1,y_2)=f(0,1/2)=0$. , $f(x_1,y_3)=f(0,1)=0$

$f(x_2,y_1)=f(1/2,0)=0$. $f(x_2,y_2)=f(1/2,1/2)=\frac{125}{64}$, $f(x_2,y_3)=f(1/2,1)=\frac{25}{4}$

$f(x_3,y_1)=f(1,0)=0$. $f(x_3,y_2)=f(1,1/2)=\frac{85}{8}$, $f(x_3,y_3)=f(1,1)=34$

In our case, the weight is

$$W = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 1 \end{bmatrix}.$$

$$S_{simpson} = \frac{(1-0)(1-0)}{9\times2\times2} \times \Big[w_{11}\times f(0,0) + w_{12}\times f(0,\tfrac{1}{2}) + w_{13}\times f(0,1) + w_{21}\times f(\tfrac{1}{2},0)$$

$$+ w_{22}\times f(\tfrac{1}{2},\tfrac{1}{2}) + w_{23}\times f(\tfrac{1}{2},1) + w_{31}\times f(1,0) + w_{32}\times f(1,\tfrac{1}{2})$$

$$+ w_{33}\times f(1,1) \Big]$$

$$= \frac{1}{36}\times\left[16\times\frac{125}{64} + 4\times\frac{25}{4} + 4\times\frac{85}{8} + 1\times34\right]$$

$$= \frac{59}{16}.$$

Accurate Solution.

$$S_{accurate} = \int_0^1 \left[\int_0^1 (9x^3 + 8x^2)\times(y^3+y)\,dx\right]dy$$

$$= \int_0^1 \left[\frac{9}{4}x^4 + \frac{8}{3}x^3\right]_0^1 \times (y^3+y)\,dy$$

$$= \left(\frac{9}{4}+\frac{8}{3}\right)\times\left[\frac{1}{4}y^4 + \frac{1}{2}y^2\right]_0^1$$

$$= \left(\frac{9}{4}+\frac{8}{3}\right)\times\frac{3}{4} = \frac{59}{16}.$$

As $S_{simpson} = S_{accurate}$, the simpson's rule is getting the accurate result. It's proved ~~before~~ ~~that~~ simpson is capable to intergrate accurately the polynomical of order 3 ~~for both directions~~. And for the double integrals the order of x and y are order of 3 at each directions. The simpson's rule performed as perdicted, given the accurate integration ~~by~~ of the problem.