Numerical Methods for Partial Differential Equation

Exercise-3

Finite Differences

By

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2)

Given PDE,

$$u_t + a u_{xx} = 0, \quad x \in (0,1), \; t \geq 0, \; a > 0$$

With initial conditions,

$$u(x,0) = \sin(2\pi x)$$

Periodic boundary conditions,

$$u(0,t) = u(1,t)$$

a) Considering BTFS, (Backward in Time Forward in Space)

We have for 3 points,

n+1	△	△
n	□	
	i	i+1

□ → Known point

△ → Unknown point

O → Where method is imposed.

The Taylor series is given by,

$$f_{i+1} = f_i + \frac{\Delta x}{1!} f_i' + O(\Delta x)^2$$

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$$\frac{df}{dx}\Big|_i = \frac{f_{i+1}-f_i}{\Delta x} - O(\Delta x)$$

$$\frac{du}{dx}\Big|_i^{n+1} = \frac{u_{i+1}^{n+1}-u_i^{n+1}}{\Delta x} - O(\Delta x)$$

Backward difference is given by

$$u\Big|_i^{n+1} = u\Big|_i^{n+1} - \Delta t\,\frac{\partial u}{\partial t}\Big|_i^{n+1} + O(\Delta t^2)$$

$$\frac{\partial u}{\partial t}\Big|_i^{n+1} = \frac{u_i^{n+1}-u_i^{n}}{\Delta t} + O(\Delta t)$$

We have FDE as, (BTFS)

$$\frac{u_i^{n+1}-u_i^{n}}{\Delta t} = a\left(\frac{u_{i+1}^{n+1}-u_i^{n+1}}{\Delta x}\right) - O(\Delta t, \Delta x)$$

$\Rightarrow (1-r)\,u_i^{n+1} + r\,u_{i+1}^{n+1} = u_i^{n}$, where $r = \frac{a\Delta t}{\Delta x}$

The truncation error is first order in both space and time. We have scheme as, $\cancel{\boxed{u_i^{n+1} + r\left(u_i^{n+1} - u_{i+1}^{n+1}\right) = u_i \quad (\text{where } r = \frac{a\Delta t}{\Delta x})}}$

The implicit method is stable (unconditionally). Hence, both Forward difference and Backward difference can be equally stable and don't make difference in solution. Thus, Forward difference is choosed for spatial derivative.

b) Periodic boundary condition is treated as a linear equation in addition to ~~the get~~ solution system of equations, to get numbers of equations equal to numbers of unknowns.

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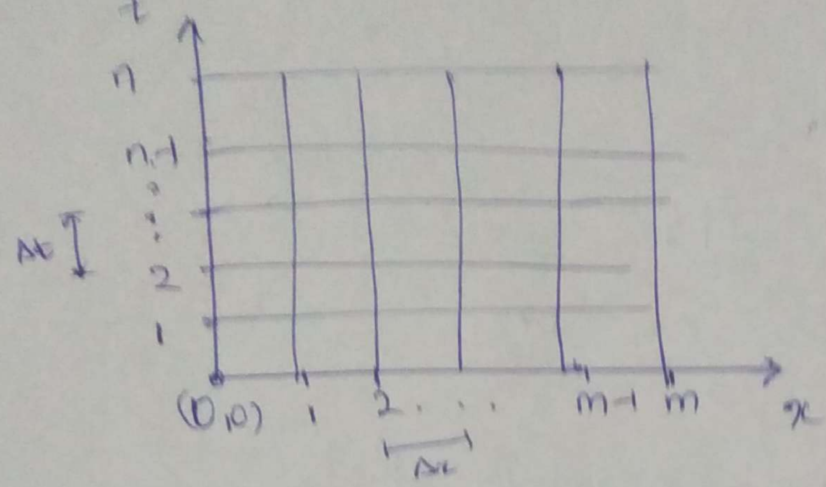
Considers, the following discretization.



~~x~~ $= x_0 + ih$, $i = 0, 1, \dots m$

~~$t = t_0 + nh$, $n = 0, 1, \dots n$~~

$t = t_0 + jh$, $j = 0, 1, 2, \dots n$

The detailed systems of equations to be in each step

is,

$i = 0$, ⇒ $(1-r)u_0^{n+1} + r u_1^{n+1} = u_0^n$ — (1)

$i = 1$. ⇒ $(1-r)u_1^{n+1} + r u_2^{n+1} = u_1^n$ — (2)

⋮

$i = m-2$ ⇒ $(1-r)u_{m-2}^{n+1} + r u_{m-1}^{n+1} = u_{m-2}^{n+1}$ — (m-2)

$i = m-1$ ⇒ $(1-r)u_{m-1}^{n+1} + r u_m^{n+1} = u_{m-1}^{n+1}$ (m-1)

$\Rightarrow (1-r)u_{m-1}^{n+1} + r u_0^{n+1} = u_{m-1}^{n+1}$ [from B.C]

At $i = m-1$ the equation u_m^{n+1} is eliminated using Boundary conditions.

c) Here, ~~Thus~~ we have m-1 unknowns with m-1 linear equations. Hence, it can be solved by Crout method (direct) and Gauss-Siedel methods (iterative)

d) Writing in matrix form,

$$Au^{n+1} = Iu^n + F$$

where.

The Fill in matrix A is given by,

(2)

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$$A = \begin{bmatrix} (1-r) & r & 0 & \cdots & & \\ 0 & (1-r) & r & & & \vdots \\ \vdots & & \ddots & & & \vdots \\ & & & & & \\ & & & (1-r) & r & \\ r & \cdots & & & & 1-r \end{bmatrix}$$

U^{n+1}, U^n are solution values at t^{n+1} and t^n

I is identity matrix.

$F =$ Zero matrix in this case. (It can be neglected)

$$U^{n+1} = \left(U_0^{n+1}, U_1^{n+1}, \ldots \, U_{m-1}^{n+1}\right)^T$$

$$U^n = \left(U_0^n, U_1^n, \ldots \, U_{m-1}^n\right)^T$$

A can be decomposed as following to solve for U^{n+1},

$\underline{A = LU}$.

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4)

Given PDE is, (diffusion-reaction PDE)

$$u_t = \nu u_{xx} + \sigma u \quad \text{in } x \in (0,1),\ t > 0$$

with B.C

$$u(0,t) = 0 \quad \text{and} \quad u_x(1,t) = 0.$$

and Initial conditions are,

$$u(x,0) = \begin{cases} 0 & \text{for } x < 1/4 \\ 4x-1 & \text{for } 1/4 \le x < 1/2 \\ -4x+3 & \text{for } 1/2 \le x \le 3/4 \\ 0 & \text{for } 3/4 < x. \end{cases}$$

$\nu > 0$ and $\sigma < 0$ are constants.

ν ↓ diffusion co-eff.

σ ↓ reaction co efficient

a)
$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + \sigma u$$

Using FTCS method,

n+1 Δ

n $\square$ $\square$ $\square$

i-1   i   i+1

$$\frac{\partial u}{\partial t} = \frac{u^{n+1} - u^n}{\Delta t} + O(\Delta t)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{(\Delta x)^2} + O(\Delta x^2)$$

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The FDE is,

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = \nu \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2} + \sigma u_i^n + \tau_i(\Delta t, \Delta x^2)$$

Simplifying

$$u_i^{n+1} = \frac{\nu\Delta t}{\Delta x^2}\left(u_{i-1} - 2u_i^n + u_{i+1}^n\right) + \sigma \Delta t\, u_i^n + u_i^n$$

Let $\frac{\nu \Delta t}{\Delta x^2} = r$, $\sigma\Delta t = \alpha$.

$$\Rightarrow \quad u_i^{n+1} = r\left(u_{i-1} - 2u_i^n + u_{i+1}^n\right) + \alpha u_i^n + u_i^n \quad \text{--- (1)}$$

$$\boxed{u_i^{n+1} = r u_{i-1}^n + (1+\alpha-2r) u_i^n + r u_{i+1}^n} \quad \text{For } i = 0, 1, 2, \dots m-1$$

With, Initial conditions $u(x,0)$ as defined before.
One of the Boundary conditions $u(0,t)=0$. [Dirichlet]
And Neumann Boundary conditions are treated as below,

for $i=m$ we have,

using BC, $$\left.\frac{\partial u}{\partial x}\right|_m^n = \frac{u_{m+1}-u_{m-1}}{2\Delta x} = 0$$

[u_{m+1} is ghost node.
$\underset{m-1}{o}$ $\underset{m}{o}$ $\underset{m+1}{\Delta}$]

$$\Rightarrow \quad \underline{\underline{u_{m+1} = u_{m-1}}}$$

substituting in FTCS equation, we have

$$u_m^{n+1} = r u_{m-1} + (1+\alpha-2r) u_m^n + r u_{m-1}^n$$

$$\underline{\underline{u_m^{n+1} = 2r u_{m-1}^n + (1+\alpha-2r) u_m^n}}$$

This can be solved for u_m^{n+1}.

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b) For $\sigma = 0$ (diffusion equation), i.e $u_t = \nu u_{xx}$.

$\Rightarrow \alpha = 0$, The FTCS scheme is,

$$\boxed{u_i^{n+1} = r u_{i-1} + (1-2r) u_i^n + r u_{i+1}^n}$$

For $\nu = 0$,

Reaction equation, $u_t = \sigma u$,

the FTCS scheme can be obtained by substituting $r = 0$ in ~~FTCS~~ FTCS equation 1,

$$\boxed{u_i^{n+1} = (1+\alpha) u_i^n}$$

c) Given, $\nu = 0.1$, $\sigma = -0.1$, $\Delta x = 0.25$ and $\Delta t = 0.1$,

and we have initial conditions,

$u_0^0 = 0$

$u_1^0 = u(0.25, 0) = 4x - 1 = 4 \times 0.25 - 1 = 0$

$u_2^0 = u(0.5, 0) = -4x + 3 = -4 \times 0.5 + 3 = 1$

$u_3^0 = u(0.75, 0) = 0$

$u_4^0 = u(1, 0) = 0$

$\bar{u}^0 = (0 \quad 0 \quad 1 \quad 0 \quad 0)^T$

We have for, 1st step,

$n = 0$,

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The FCT

$$r = \frac{\nu \Delta t}{(\Delta x)^2} = \frac{0.1 \times 0.10}{(0.25)^2} = \underline{0.16}$$

$$\alpha = \sigma \Delta t = -0.1 \times 0.1 = \underline{-0.01}$$

$u_0^1 = 0$

for $i=1$,

$$u_1^1 = r u_0^0 + (1+\alpha-2r) u_1^0 + r u_2^0$$

$$= (+0.160 \times 0) + (1+\alpha-2r) \times 0 + (+0.160 \times 1)$$

$$= \underline{+0.160}$$

We have similarly,

$$u_2^1 = r u_1^0 + (1+\alpha-2r) u_2^0 + r u_3^0 = (1+\alpha-2r) 1 = 0.67$$

$$u_3^1 = r u_2^0 + (1+\alpha-2r) u_3^0 + r u_4^0 = r u_2^0 = +0.160 \times 1 = 0.16$$

$$u_4^1 = r u_3^0 + (1+\alpha-2r) u_4^0 + r u_5^0 = 0$$

$$\bar{u}^1 = (0 \quad 0.16 \quad 0.67 \quad 0.16 \quad 0)^T$$

For second step,

$u_0^2 = 0$

$$u_1^2 = r u_0^1 + (1+\alpha-2r) u_1^1 + r u_2^1 = 0.67 \times 0.16 + 0.16 \times 0.67 = 0.2144$$

$$u_2^2 = 0.16 [u_1^1 + u_3^1] + (1+\alpha-2r) u_2^1 = 0.16 [0.16 + 0.16] + 0.67 \times 0.67 = 0.5001$$

$$u_3^2 = 0.16 [u_2^1 + u_4^1] + 0.67 u_3^1 = 0.16 [0.67] + 0.67 \times 0.16 = 0.2144$$

$$u_4^2 = 0.16 [2 \times u_3^1] + 0.67 u_4^1 = 0.16 [2 \times 0.16] + 0 = 0.0512$$

$$\bar{u}^2 = (0 \quad 0.2144 \quad 0.5001 \quad 0.2144 \quad 0.0512)^T$$

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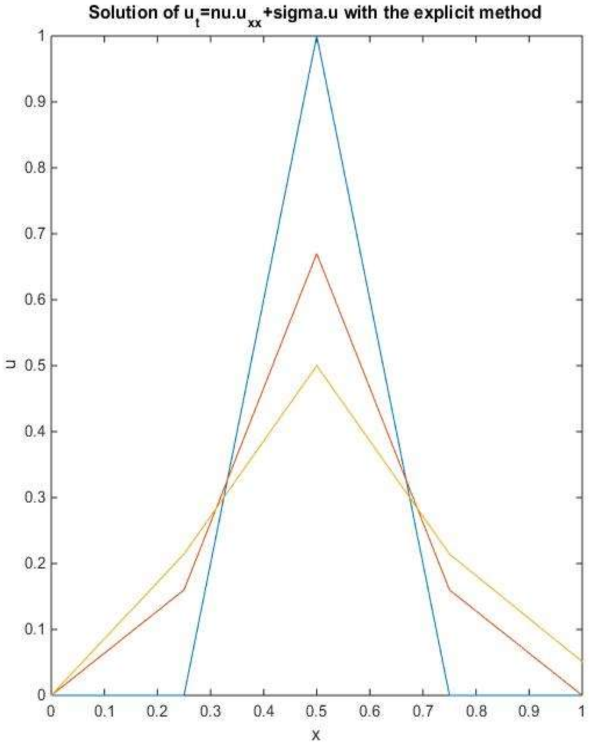

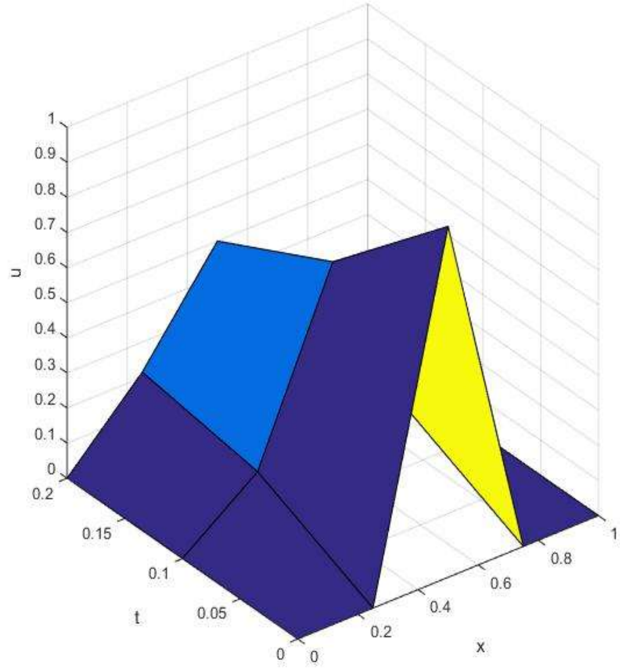

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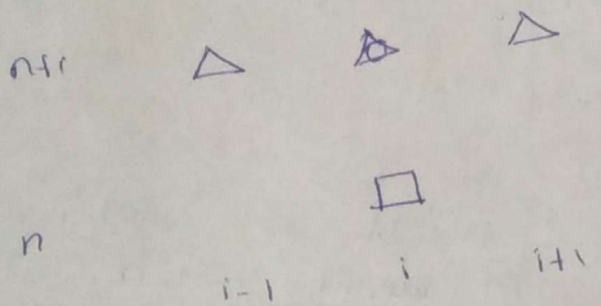
From the graphic profile of u the obtained results seems reasonable. The symmetry of solution is disturbed due to different type of boundary conditions [i.e Neuman at $x=1$ and Dirichlet at $x=0$]. We can visualize the diffusion through graph.

d) Implicit scheme,

Consider BTCS method,



$$\left.\frac{\partial u}{\partial t}\right|_i^{n+1} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_i^{n+1} \approx \frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{\Delta x^2}$$

substituting diffusion-reaction equation

$$u_t = \nu u_{xx} + \sigma u$$

$$\Rightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu \left(\frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{\Delta x^2} \right) + \sigma u_i^{n+1}$$

$$u_i^{n+1} - u_i^n = r\left(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}\right) + \alpha u_i^{n+1}$$

Where, $r = \frac{\nu \Delta t}{\Delta x^2}$, $\alpha = \sigma \Delta t$

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$$-r u_{i-1}^{n+1} + (1+2r-d)\, u_i^{n+1} - r u_{i+1}^{n+1} = u_i^n$$

with ~~boundary~~ initial conditions $u(x,0)$ $\boxed{\Rightarrow \cancel{u^0} = 0}$

and Dirichlet Boundary condition $u(0,t) = 0 \Rightarrow \bar{u}_0^n = 0$ for $t > 0$

For Neumann boundary condition,

$u_x(1,t) = 0$ for $t > 0$

we have, using central differences,

$$\left.\frac{\partial u}{\partial t}\right|_m^{n+1} = \frac{u_{m+1}^{n+1} - u_{m-1}}{2\Delta x} = 0$$

[$\cdots$ o o Δ
$m-1$ m $m+1$
$\downarrow$
ghost node]

$$\Rightarrow \underline{u_{m+1}^{n+1} = u_{m-1}^{n+1}}$$

Writing in matrix form, we have

$$A u^{n+1} = I u^n$$

| u^{n+1}, u^n are solution matrices
| I identity matrix.

The equations for a time step are,

$$-r(\cancel{u_0^{n+1}})^{\to 0} + (1+2r-d)\, u_1^{n+1} - r u_{1+1}^{n+1} = u_1^n \quad \text{— (1)}$$

$$-r(u_1^{n+1}) + (1+2r-d)\, u_2^{n+1} - r u_3^{n+1} = u_2^n \quad \text{— (2)}$$

$$\vdots$$

$$-r u_{m-2}^{n+1} + (1+2r-d)\, u_{m-1}^{n+1} - r u_m^{n+1} = u_{m-1}^n \quad \text{— (m-1)}$$

$$2r u_{m-1}^{n+1} + (1+2r-d)\, u_m^{n+1} - 0 = u_m^n \quad \text{— (m)}$$

The matrix A is of the structure,

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$$A = \begin{bmatrix} 1+2r-\alpha & -r & & & \\ -r & 1+2r-\alpha & -r & & \\ & \ddots & \ddots & \ddots & \\ & & -r & 1+2r-\alpha & -r \\ & & & -2r & 1+2r-\alpha \end{bmatrix}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

Since, matrix A is diagonally banded and has 3 lines in band, we can solve this system of linear equations as tri-diagonal system with Thomas Algorithm. This is most suitable method. ~~We can also~~ ~~x~~ The other good method is decomposition method.

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