Numerical Methods for Partial Differential Equations
Homework- Ordinary Differential Equations
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Exercise-1

A.1 To solve the following second-order ODE, first it must be reduced to a system of first order ODEs.

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

$$t \in [0.1] \quad L = 1 \quad g = 9.81 \quad \theta(1) = 0.4 \quad \frac{d\theta}{dt}(1) = 0$$

$$\theta(t)_{exact} = \frac{2}{5}\sin\sqrt{\frac{g}{L}}\sin\sqrt{\frac{gt}{L}} + \frac{2}{5}\cos\sqrt{\frac{g}{L}}\cos\sqrt{\frac{gt}{L}}$$

$$\boldsymbol{\theta} = \begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}\theta\\ \frac{d\theta}{dt}\end{Bmatrix}, \qquad \frac{d\boldsymbol{\theta}}{\boldsymbol{dt}} = \begin{Bmatrix}\frac{d\theta_1}{dt}\\ \frac{d\theta_2}{dt}\end{Bmatrix} \qquad \boldsymbol{f}(\boldsymbol{\theta},t) = \begin{Bmatrix}\theta_2\\ -\frac{g}{L}\theta_1\end{Bmatrix} \qquad \boldsymbol{\alpha}(1) = \begin{Bmatrix}0.4\\ 0\end{Bmatrix}$$

Then, Second-order Runge-Kutta method for the above system will be:

$$\theta_1^{i+1} = \theta_1^i + \frac{h}{2}(k_{11} + k_{12}), \qquad \theta_2^{i+1} = \theta_2^i + \frac{h}{2}(k_{21} + k_{22})$$

Where

$$k_{11} = f_1(t^i, \theta_1^i, \theta_2^i) \qquad k_{12} = f_1(t^i, \theta_1^i + k_{11}h, \theta_2^i + k_{21}h)$$

$$k_{21} = f_2(t^i, \theta_1^i, \theta_2^i) \qquad k_{22} = f_2(t^i, \theta_1^i + k_{11}h, \theta_2^i + k_{21}h)$$

A.1.1 To obtain the $\theta(0)$ with 2 time steps, the starting point of iteration was $t = 1$ with $h = -0.5$

Time	$d\theta/dt \;\; or \;\; \theta_2$	θ or θ_1	θ_{exact}
1	0.00000	0.40000	0.40000
0.5	1.96200	-0.09050	0.00190
0	-0.8878	**-0.96052**	-0.39998

A.1.2 To obtain the $\theta(0)$ with 4 time steps, the starting point of iteration was $t = 1$ with $h = -0.2$

Time	$d\theta/dt \;\; or \;\; \theta_2$	θ or θ_1	θ_{exact}
1	0.00000	0.40000	0.40000
0.75	0.98100	0.27738	0.00190
0.5	1.36052	-0.05291	0.28351
0.25	0.81368	-0.37682	-0.28082
0	-0.35991	**-0.46472**	-0.39998

B.1 In order to compute an approximation of the relative error in the two step solution, it was compared with the values in the four step solution.

$$E_{relative} = \left|\frac{\theta(0)_{h=0.5} - \theta(0)_{h=0.25}}{\theta(0)_{h=0.5}}\right| = \left|\frac{-0.96052 + 0.46472}{-0.96052}\right| = \mathbf{0.51617966}$$

C.1 Since the required h_{new} is with a relative error three orders of magnitude smaller, it can be said:

$$Tol = E_r \times 10^{-3}$$

Taking into account the used method is a second-order, with $O(h^3)$ local error, the new step can be obtained as following:

$$E_h = O(h^3)\,, \qquad Tol = O(h_{new}^3)$$

$$h_{new} = \left(\frac{Tol}{E_r}\right)^{\frac{1}{3}} \times h = \left(\frac{E_r \times 10^{-3}}{E_r}\right)^{\frac{1}{3}} \times h = \frac{\boldsymbol{h}}{\mathbf{10}}$$

Exercise-2

$$\frac{dy}{dx} = y - x^2 + 1 \qquad x \in [0.1] \qquad y(0) = 1$$

$$y_{eacxt} = x^2 + 2x + 1$$

A.2 Euler method was applied using the following equation

$$Y_{i+1} = Y_i + f(x_i, Y_i)h$$

With $h = 0.25$ and starting values($Y_0 = 1,$ & $x_0 = 0$), the values in each step were:

x_i	Y_i	Y_{eacxt}
0	1.00000	1.00000
0.25	1.50000	1.56250
0.5	2.10938	2.25000
0.75	2.82422	3.06250
1	**3.63965**	4.00000

B.2 The cost of each method is considered as the number of evaluations of slop $f(x, y)$. The step size for Heun method with the same cost as Euler method in the previous section is $\boldsymbol{h_{Huen}} = \mathbf{0.5}$, and it was found as following:

$$cost_{Euler} = 1/step \qquad cost_{Huen} = 2\ /step$$

Then

$$The\ Cost = cost_{euler} \times n_{euler} = 4 \times 1 = 4$$

$$n_{Huen} = \frac{Cost}{cost_{huen}} = \frac{4}{2} = 2 \quad \rightarrow \quad h_{Huen} = \frac{ODE_{interval}}{n_{Huen}} = \frac{1}{2} = 0.5$$

Then, Huen method was applied using the following equation

$$Y_{i+1} = Y_i + \frac{h}{2}\ (k_1 + k_2)$$

Where: $k_1 = f(x_i, Y_i)$ $k_2 = f(x_i, Y_i + k_1 h)$

With $h = 0.5$ and starting values($Y_0 = 1,\ \&\ x_0 = 0$), the values in each step were:

x_i	Y_i	Y_{eacxt}
0	1.00000	1.00000
0.5	2.18750	2.25000
1	**3.83594**	4.00000

C.2 To compute the pure interpolation polynomial that fits the results in B, a second-order polynomial is used.

$$y = a_0 + a_1 x + a_2 x^2$$

$$\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0.25 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2.18750 \\ 3.83594 \end{bmatrix} \quad \rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.91406 \\ 0.92188 \end{bmatrix}$$

$$\boldsymbol{y = 1 + 1.91406\ \ x + 0.92188\, x^2}$$

D2. To approximate the results in obtained in A2 a reasonable choice will be to use Least Squares Fitting with second-order polynomial, because the available information about the solution indicates that it is a polynomial of second order.

$$y = c_0 + c_1 x + c_2 x^2$$

$$\begin{bmatrix} 5 & \sum x & \sum x^2 \\ \sum x & \sum x^2 & \sum x^3 \\ \sum x^2 & \sum x^3 & \sum x^4 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum xy \\ \sum x^2 y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2.5 & 1.875 \\ 2.5 & 1.875 & 1.5625 \\ 1.875 & 1.5625 & 1.3828125 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 11.073242 \\ 7.1875 \\ 5.84936523 \end{bmatrix}$$

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0.999135 \\ 1.799888 \\ 0.841518 \end{bmatrix}$$

$$\boldsymbol{y = 0.99913 + 1.79988x + 0.841518x^2}$$

It can be seen that Least squares Fitting produced the same polynomial as the one in C2 with slight difference in the constant values, However if both of them were compared to the analytical solution ($y = x^2 + 2x + 1$) it can be said that the first polynomial with less would describe the solution more accurately.

Exercise-3

A3. Firstly, the solution in Forward Euler method can be represented by a Taylor series expansion about a the value (x_i, y_i)

$$y_{i+1} = y_i + y_i' h + \frac{y_i''}{2!} h^2 + \cdots + \frac{y_i^{(n)}}{n!} h^n + R_n$$

$$y_{i+1} = y_i + f'(x_i, y_i) h + \frac{f''(x_i, y_i)}{2!} h^2 + \cdots + \frac{f^{(n)}(x_i, y_i)}{n!} h^n + R_n$$

$$y_{i+1} = \underbrace{y_i + f'(x_i, y_i) h}_{Euler\ \text{Approximation}} + \underbrace{\frac{f''(x_i, y_i)}{2!} h^2 + \cdots + \frac{f^{(n)}(x_i, y_i)}{n!} h^n + R_n}_{Local\ Error\ or\ Residual\ (R)}$$

For small values of h the local error can be represented by the first term only, neglecting other terms.

$$R(h^2) = \frac{f''(x_i, y_i)}{2!} h^2 = O(h^2)$$

Therefore, Forward Euler method can be written:

$$y_{i+1} = y_i + f'(x_i, y_i) h + O(h^2)$$

$$f'(x_i, y_i) = \frac{y_{i+1} - y_i}{h} - \left\{\frac{O(h^2)}{h}\right\} \rightarrow Truncation\ error\ (\tau(h))$$

Yes!! , from the above expression it can be said that the method is consistent, because the truncation error is proportional to h which means that when h goes to zero the error also goes to zero.

$$(\tau(h) \rightarrow 0 \quad when \quad h \rightarrow 0)$$

B3. For given ordinary differential equation as

$$\frac{dy}{dx} = f(x, y)$$

$$x \in [0.1] \qquad y(0) = 1$$

Backward Euler method for integration can be stated as following:

$$Y_{i+1} - Y_i - f(x_{i+1}, Y_{i+1}) h = 0$$

C3. The stability limits is found by putting the absolute value of the amplification factor less than one.
Given model equation:

$$\frac{dy}{dx} = \lambda y$$

For Forward Euler method:

$$Y_{i+1} = Y_i + Y_i \lambda \ h = Y_i \ \overbrace{(1+\lambda h)}^{\substack{amplification \\ Factor\ G}}$$

$$|G| < 1 \quad \rightarrow \quad |1+\lambda h| < 1 \quad \rightarrow \quad \boldsymbol{-2 < \lambda h < 0}$$

For backward Euler method

$$Y_{i+1} = Y_i + Y_{i+1}\lambda h \rightarrow Y_{i+1} = \overbrace{\frac{1}{(1-\lambda h)}}^{\substack{amplification \\ Factor\ G}} \ Y_i$$

$$|G| < 1 \quad \rightarrow \left|\frac{1}{(1-\lambda h)}\right| < 1 \rightarrow \quad |1-\lambda h| > 1$$

$$\boldsymbol{\lambda h < 0} \quad \& \quad \boldsymbol{\lambda h > 2}$$

D3. Given model equation

$$\frac{dy}{dx} = f(x,y) = -25\, y^{3.5}$$

Backward Euler method was applied using the following equation

$$Y_{i+1} - Y_i - f(x_{i+1}, Y_{i+1})h = 0$$

$$Y_{i+1} - Y_i + 25\ Y_{i+1}^{3.5}\ h = 0$$

And in order to apply Newton method, the derivative of above equation:

$$1 + 87.5\, {Y_{i+1}}^{2.5}\ h = 0$$

And, then Newton method in each step with two iterations can be stated as following

$$Y_{i+1}^{j+1} = Y_{i+1}^{j} - \frac{Y_{i+1}^{j} - Y_i + 25\left(Y_{i+1}^{j}\right)^{3.5} \mathrm{h}}{1 + 87.5\left(Y_{i+1}^{j}\right)^{2.5} h} \qquad j = 0,1,2$$

Where Y_i is chosen as initial guess in each step ($Y_{i+1}^{0} = Y_i$).

With $h = 0.1$ and starting values($Y_0 = 1,\ \&\ x_0 = 0$), the values in each step were:

x_i	Y_i	Y_{eacxt}
0	1.00000	1.00000
0.1	0.62179	0.45276
0.2	0.46025	0.35307

E 3. Euler method was applied using the following equation

$$Y_{i+1} = Y_i + f(x_i, Y_i)h$$

With $h = 0.1$ and starting values($Y_0 = 1,\ \&\ x_0 = 0$), the values in each step were

x_i	Y_i	Y_{eacxt}
0	1.00000	1.00000
0.1	-1.50000	0.45276
0.2	-1.5+10.33378i	0.35307

F 3. In order to find the stability condition in Euler method for the ODE: $\frac{dy}{dx} = -25\ y^{3.5}$ must be linearized first.

Using Taylor Series at $y = 1$, for $f(y) = -25\ y^{3.5}$

$$f(y) = f(1) + f'(1)\ (y-1) + \cdots + \frac{f^n(1)}{n!}\ (y-n)^n + \cdots$$

Taking only the liner terms

$$f(y) = 62.5 - 87.5\ y$$

Then, Euler method can written as

$$Y_{i+1} = Y_i + (62.5 - 87.5\ Y_i)h = Y_i\ \overbrace{(1 - 87.5h\ )}^{\substack{amplification \\ Factor\ G}} + 62.5$$

$$|G| < 1 \quad \rightarrow \quad |1 - 87.5h| < 1 \quad \rightarrow \quad \boldsymbol{0 < h < \frac{2}{87.2}}$$

Then, Matlab code was used to check the stability of the given step sizes, and the result was:

- For $h = \frac{1}{10}$ & $h = \frac{1}{15}$ the results fitted the stability condition, because in both cases the value of y became minus after one step, and in the second it became complex number.
- As for $h = \frac{1}{30}$, the stability condition did not hold because there were no minus values or complex number for y, and by increasing the number of steps it converge.
- For $h = \frac{1}{45}$ & $h = \frac{1}{90}$ the results fitted the criteria in the stability condition, and the solution was stable and smooth.

From this analysis, it can be stated that the maximum possible step size is $\boldsymbol{\frac{1}{15}}$

Exercise-4

A.4 For the second-order ordinary equation

$$\frac{d^2y}{dx^2} + \omega^2 y = 0$$

$$x \in [0.1] \qquad y(0) = 0 \qquad \frac{dy}{dt}(0) = \omega$$

$$y(x)_{exact} = \sin(\omega x)$$

The reduced system of first order ODEs is:

$$\boldsymbol{y} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} y \\ \frac{dy}{dt} \end{Bmatrix}, \qquad \frac{d\boldsymbol{y}}{\boldsymbol{dt}} = \begin{Bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{Bmatrix} \qquad \boldsymbol{f}(\boldsymbol{y}, x) = \begin{Bmatrix} y_2 \\ -\omega^2 y_1 \end{Bmatrix} \qquad \boldsymbol{\alpha}(1) = \begin{Bmatrix} 0 \\ \omega \end{Bmatrix}$$

B.4 Then, The Forward Euler method for the above system will be:

$$Y_1^{i+1} = Y_1^i + f_1(x_i, Y_1^i, Y_2^i)h \qquad\qquad Y_2^{i+1} = Y_2^i + f_2(x_i, Y_1^i, Y_2^i)h$$

To obtain the $y_1(1)$ & $y_2(1)$ with 4 time steps, the starting point of iteration was $x = 0$,$y_1(0) = 0$ $y_2(0) = \omega = 3$ with $h = 0.25$

x	$dy/dx \quad or \quad y_2$	y or y_1	y_{exact}
0	3.00000	0.00000	0.00000
0.25	3.00000	0.75000	0.68164
0.5	1.31250	1.50000	0.99749
0.75	-2.06250	1.82813	0.77807
1	-6.17578	**1.31250**	0.14112

C.4 Then, the solution with 8 time steps was:

x	$dy/dx \quad or \quad y_2$	y or y_1	y_{exact}
0.000	3.00000	0.00000	0.00000
0.125	3.00000	0.37500	0.36627
0.25	2.57813	0.75000	0.68164
0.375	1.73438	1.07227	0.90227
0.5	0.52808	1.28906	0.99749
0.625	-0.92212	1.35507	0.95409
0.75000	-2.44658	1.23981	0.77807
0.87500	-3.84136	0.93399	0.49392
1.00000	-4.89209	**0.45382**	0.14112

In order to compute an approximation of the relative in the 8 steps solution, the solution will be compared with the value in the four step solution.

$$E_{relative} = \left|\frac{y(1)_{h=0.125} - \theta(1)_{h=0.25}}{\theta(1)_{h=0.125}}\right| = \left|\frac{0.45382 - 1.31250}{0.45382}\right| = 1.892115817$$

The required step size is to obtain a numerical solution with three significative digits, so it can be said:

$$Tol = 10^{-4}$$

And since the used method is a first order method, with $O(h^2)$ local error, the new step can be obtained as following:

$$E_h = O(h^2)\,, \qquad Tol = O(h_{new}^2)$$

$$h_{new} = \left(\frac{Tol}{E_r}\right)^{\frac{1}{2}} \times h = \left(\frac{10^{-4}}{1.892115817}\right)^{\frac{1}{2}} \times 0.125 = 9.08732697 \times 10^{-4}$$

$$n = \frac{1}{9.08732697} \times 10^4 = 1100.43 \cong 1100$$

$$\boldsymbol{h_{new}} = \frac{\mathbf{1}}{\mathbf{1100}}$$

With the new step $\boldsymbol{y(1)} =$**0.14170589**, where the exact is $\boldsymbol{y_{exact}(1)} = \mathbf{0.14112.}$