EXERCISES | GABRIEL VALDÉS ALONSO

1) COMPUTE THE UNIQUE REAL ROOT OF $F(x) = x^3 + 2x^2 + 10x - 20 = 0$ WITH 4 ITERATIONS OF NEWTON'S METHOD WITH THE INITIAL APPROXIMATION $x^0 = \sqrt[3]{20}$. PLOT THE CONVERGENCE GRAPHIC. DOES NEWTON'S METHOD BEHAVE AS EXPECTED?

NEWTON METHOD IS USED DEFINING

$$\left.\begin{aligned} & x^{k+1} = x^k + \Delta x^{k+1} \\ & F(x^{k+1}) = F(x^k + \Delta x^{k+1}) \approx F(x^k) + F'(x^k)\,\Delta x^{k+1} \end{aligned}\right\} \Delta x^{k+1} = -\frac{F(x^k)}{F'(x^k)}$$

WE HAVE: $x^0 = \sqrt[3]{20}$

$F'(x) = 3x^2 + 4x + 10$

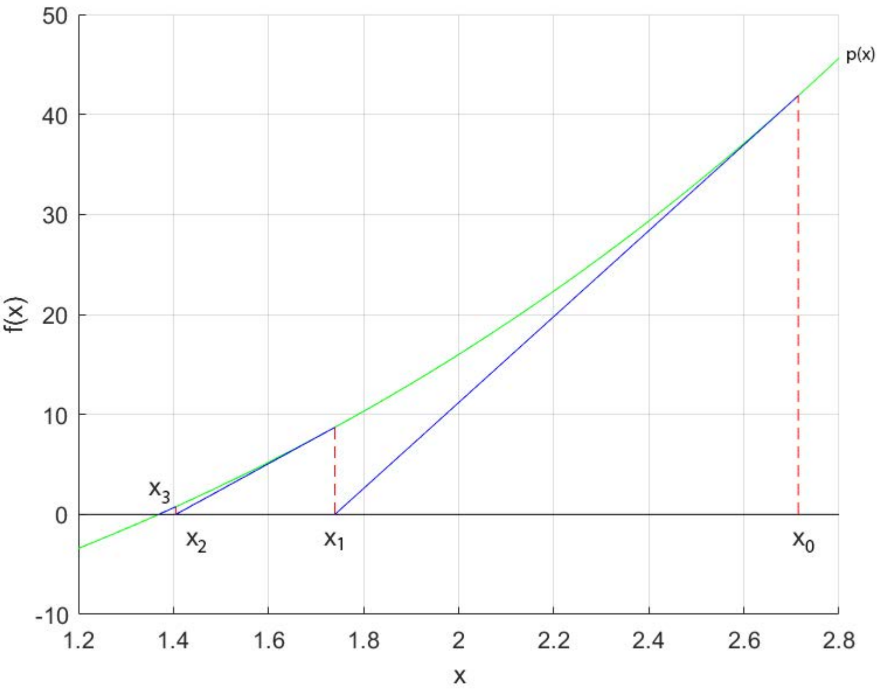
$$k=0 \rightarrow \begin{cases} \Delta x^1 = -\left(20 + 2\cdot 20^{2/3} + 10\cdot 20^{1/3} - 20\right)/\left(3\cdot 20^{2/3} + 4\cdot 20^{1/3} + 10\right) \approx -0{,}97483 \\ F(x^1) \approx (20 + 2\cdot 20^{2/3} + 10\cdot 20^{1/3} - 20) - 0{,}97483\,(3\cdot 20^{2/3} + 4\cdot 20^{1/3} + 10) \approx 2{,}07295\cdot 10^{-4} \\ x^1 = 20^{1/3} - 0{,}97483 \approx 1{,}73959 \end{cases}$$

$$k=1 \rightarrow \begin{cases} \Delta x^2 = -F(x^1)/F'(x^1) \approx -0{,}33462 \\ F(x^2) = F(x^1) + F'(x^1)\,\Delta x^2 \approx 8{,}68451\cdot 10^{-3} \\ x^2 = x^1 + \Delta x^2 \approx 1{,}40497 \end{cases}$$

$$k=2 \rightarrow \begin{cases} \Delta x^3 = -F(x^2)/F'(x^2) \approx -0{,}035787 \\ F(x^3) = F(x^2) + F'(x^2)\,\Delta x^3 \approx -4{,}024806\cdot 10^{-6} \\ x^3 = x^2 + \Delta x^3 \approx 1{,}369183 \end{cases}$$

$$k=3 \rightarrow \begin{cases} \Delta x^4 = -F(x^3)/F'(x^3) \approx -0{,}00037485 \\ F(x^4) = F(x^3) + F'(x^3)\,\Delta x^4 \approx 3{,}16718\cdot 10^{-8} \\ x^4 = x^3 + \Delta x^4 \approx 1{,}36881 \end{cases}$$

ANALYTICAL SOLUTION WITH 5 DECIMAL PRECISION. $x \approx 1{,}36881$



AS CAN BE SEEN ON THE PLOT THE METHOD WORKS AS EXPECTED, SINCE IN EVERY ITERATION THE LINES GET CLOSER TO THE POINT. ANALYSING THE NUMBERS IN THE STEPS WE SEE THAT x CONVERGES AROUND $k=2$, AND THAT $F(x^k)$ KEEPS GETTING SMALLER AT FAST RATES, THE SAME AS THE INCREMENT Δx^k.

:) IN INTERVAL (0,1) FOR A 3RD ORDER QUADRATURE

a) DETERMINE MINIMUM NUMBER OF INTEGRATION POINTS AND SPECIFY THE INTEGRATION POINTS AND WEIGHTS.

b) IS IT POSSIBLE TO OBTAIN A 3RD ORDER QUADRATURE WITH POINTS $x_0 = 1/4$, $x_1 = 1/2$, $x_2 = 3/4$ AND $x_3 = 1$? IF IT IS, COMPUTE THE WEIGHTS, IF NOT JUSTIFY.

A) IF WE USE A GAUSSIAN QUADRATURE, WE HAVE THAT $2N+1 = 3$, SO WE NEED ACTUALLY $N=1$ OR TWO POINTS. WITH ONE INTERVAL WE HAVE:

$$\rightarrow I = \int_0^1 F(x)\,dx = \frac{1}{2}\int_{-1}^{1} F\left(\frac{1}{2}z + \frac{1}{2}\right) \simeq \frac{1}{2}\left[w_0 F\left(\frac{1}{2}x_0 + \frac{1}{2}\right) + w_1\left(\frac{1}{2}x_1 + \frac{1}{2}\right)\right]$$

AND FOR $N=1$ THE WEIGHTS ARE

$$\left.\begin{array}{l} w_0 = 1 \\ w_1 = 1 \end{array}\right\} \rightarrow \begin{array}{l} w_0 = \frac{1}{2} \\ w_1 = \frac{1}{2} \end{array}$$ (APPLYING THE CHANGE OF VARIABLES)

AND THE POINTS

$$\left.\begin{array}{l} z_0 = -\sqrt{3}/3 \\ z_1 = \sqrt{3}/3 \end{array}\right\} \rightarrow \begin{array}{l} x_0 = \frac{3-\sqrt{3}}{6} \\ x_1 = \frac{3+\sqrt{3}}{6} \end{array}$$ (APPLYING CHANGE OF VARIABLES).

B)

6) A) IF N+1 POINTS GAUSSIAN QUADRATURE IS USED FOR NUMERICAL INTEGRATION STATE THE ORDER OF THE POLYNOMIAL THAT IS INTEGRATED EXACTLY.

B) IF N=2 IS SELECTED FOR GAUSSIAN QUADRATURE, WHICH (IF ANY) OF THE FOLLOWING INTEGRALS WILL BE INTEGRATED EXACTLY?

i) $\int_0^1 \sin x \, dx$ ii) $\int_0^1 x^3 \, dx$ iii) $\int_0^1 x^4 \, dx$ iv) $\int_0^1 x^{5.5} \, dx$

A) GAUSS QUADRATURES OF N INTERVALS HAVE ORDER 2N+1. CHOOSING N+1 POINTS IS EQUIVALENT TO HAVING N INTERVALS, SO THE ORDER OF THE QUADRATURE IS 2N+1.

B) FOR N=2 WE COULD INTEGRATE EXACTLY POLYNOMIAL UP TO ORDER 5. THE ERROR OF THE QUADRATURE IS

$$E_N = \Omega_N f^{(6)}(\mu)$$

APPLYING TO FUNCTIONS,

i) $(\sin x)^{(6)} = -\sin x$ (NOT INTEGRABLE EXACTLY)

ii) $(x^3)^{(6)} = 0$ (EXACT INTEGRATION)

$(x^4)^{(6)} = 0$ (EXACT INTEGRATION)

$(x^{11/2})^{(6)} = \dfrac{10395}{64\sqrt{x}}$ (NOT EXACTLY INTEGRABLE).

7) COMPUTE $\int_0^1 12x \, dx$; $\int_0^1 (5x^3 + 2x) \, dx$ BY HAND USING

i) TRAPEZOIDAL RULE OVER 2 UNIFORM INTERVALS

ii) SIMPSON'S RULE OVER 2 UNIFORM INTEGRALS

COMPUTE THE ERROR OF BOTH APPROXIMATIONS. ARE THE METHODS BEHAVING AS EXPECTED?

i) $\int_a^b f(x)\,dx \approx \frac{h}{2}(f(x_0) + f(x_1))$

USING $N=2 \to h = \frac{1}{2} \to \int_a^b f(x)\,dx \approx \frac{h}{2}\sum_{k=1}^{2}(f(x_{k+1}) + f(x_k))$

$$\int_0^1 12x \, dx \approx \frac{1}{4}\left[12\cdot\left(\tfrac{1}{2}\right) + 12\cdot 0 + 12\cdot 1 + 12\cdot\left(\tfrac{1}{2}\right)\right] = 6$$

$$\int_0^1 (5x^3+2x)\,dx \approx \frac{1}{4}\left[\left(5\cdot\tfrac{1}{8} + 2\cdot\tfrac{1}{2}\right) + 0 + (5\cdot 1 + 2\cdot 1) + \left(5\,\tfrac{1}{8} + 2\cdot\tfrac{1}{2}\right)\right] = 41/16$$

ii) $\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

USING $N=2 \to \int_a^b f(x)\,dx \approx \frac{b-a}{12}\left[f(a) + 4f\left(\frac{a+b}{4}\right) + 2f\left(\frac{a+b}{2}\right) + 4f\left(\frac{3(a-b)}{4}\right) + f(b)\right]$

$$\int_0^1 12x\,dx \approx \frac{1}{12}\left[0 + 4\cdot\left(12\cdot\tfrac{1}{4}\right) + 2\cdot\left(12\cdot\tfrac{1}{2}\right) + 4\cdot\left(12\cdot\tfrac{3}{4}\right) + (12\cdot 1)\right] = 6$$

$$\int_0^1 (5x^3+2x)\,dx \approx \frac{1}{12}\left[0 + 4\cdot\left(5\,\tfrac{1}{64} + 2\cdot\tfrac{1}{4}\right) + 2\cdot\left(5\cdot\tfrac{1}{8} + 2\cdot\tfrac{1}{2}\right) + 4\cdot\left(5\cdot\tfrac{27}{64} + 2\cdot\tfrac{3}{4}\right) + (5\cdot 1 + 2\cdot 1)\right] = 9/4$$

- ANALYTICAL SOLUTION

$$\int_0^1 12x\,dx = 12\cdot\frac{x^2}{2}\Big|_0^1 = 6$$

$$\int_0^1 (5x^3+2x)\,dx = 5\frac{x^4}{4}\Big|_0^1 + 2\frac{x^2}{2}\Big|_0^1 = 9/4$$

- ERROR

SIMPSON'S: ZERO BOTH CASES SINCE $f^{(4)} = 0$.

TRAPEZOIDAL:

$$(12x)'' = 0 \to E = 0$$
$$(5x^3+2x)'' = 30x \to E = -\frac{h^3}{12}f''(\mu) = -\frac{2\cdot 1/8}{12}\cdot 30\mu = -5/8\,\mu$$

$$\int_0^1 (5x^3+2x)\,dx = 41/16 - 5/8\,\mu = 9/4 \quad (\text{IF } \mu = 1/2)$$

THE METHODS BEHAVE AS EXPECTED. TRAPEZOID RULE GIVES NO ERROR FOR INTEGRATION OF A LINEAR FUNCTION SINCE IS A LINEAR APPROXIMATION, BUT FOR A HIGHER ORDER POLYNOMIAL EXIST SOME ERROR. IN THE CASE OF THE SIMPSON RULE IN BOTH CASES WE HAVE ZERO ERROR BECAUSE THE APPROXIMATION WORKS FLAWLESSLY FOR ANY POLINOMIAL OF ORDER 3 OR LESS.

10) PERFORM THE NUMERICAL INTEGRATION OF

$$\int_0^1 \int_0^1 (9x^3 + 8x^2)(y^3 + y) \; dx \, dy$$

USING SIMPSON'S RULE IN EACH DIRECTION. IS THE APPROXIMATION BEHAVING AS EXPECTED?

THE ANALYTICAL SOLUTION FOR THE INTEGRAL IS

$$\underbrace{\left(9\frac{x^4}{4}\Big|_0^1 + 8\frac{x^3}{3}\Big|_0^1\right)}_{59/12}\underbrace{\left(\frac{y^4}{4}\Big|_0^1 + \frac{y^2}{2}\Big|_0^1\right)}_{3/4} = \frac{59}{16} \approx 3{,}6875$$

THIS INTEGRAL CAN BE SEPARATED, SO WE APPLY SIMPSON RULE INDEPENDENTLY IN EACH DIRECTION

$$\int_a^b F(x)\,dx \approx \frac{b-a}{6}\left[F(a) + 4F\left(\frac{a+b}{2}\right) + F(b)\right]$$

USING ONE INTERVAL:

$$\left.\begin{aligned} \int_0^1 (9x^3 + 8x^2)\,dx &\approx \frac{1}{6}\left[0 + 4\left(9\cdot\frac{1}{8} + 8\cdot\frac{1}{4}\right) + (9+8)\right] = 59/12 \\ \int_0^1 (y^3 + y)\,dy &\approx \frac{1}{6}\left[0 + 4\left(\frac{1}{8} + \frac{1}{2}\right) + (1+1)\right] = 3/4 \end{aligned}\right\} \; 59/16 \approx 3{,}6875$$

THE APPROXIMATION IS THEN EXACT AND WE DON'T NEED TO USE FURTHER INTERVALS. THIS BEHAVIOR IS EXPECTED BECAUSE THE ERROR IN SIMPSON'S RULE DEPENDS ON THE 4TH DERIVATIVE OF THE FUNCTION TO BE INTEGRATED. SINCE OUR POLYNOMIALS ARE OF ORDER 3, WE HAVE 0 AS A FOURTH DERIVATIVE, WHICH GIVES THE ZERO ERROR.