NUMERICAL METHODS FOR PDE

List of Exercises N°2

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1. The motion of a non-frictional pendulum is governed by the Ordinary Differential Equation (ODE)

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

where θ is the angular displacement, $L = 1m$ is the pendulum length and the gravity acceleration is $g = 9.8m/s^2$.
The position and velocity at time $t = 1s$ are known:

$$\theta(1) = 0.4\ rad \quad ; \quad \frac{d\theta}{dt}(1) = 0\ rad/s$$

a) Solve the initial boundary value problem in the interval $(0, 1)$ using a second-order Runge-Kutta method to determine the initial position at $t = 0s$, with 2 and 4 time steps.

To use the Runge-Kutta method, the first thing to do is write the ODE as a linear system of equations. We define $\theta = \theta_1$ and $d\theta_1/dt = \theta_2$, where we get the vector $\boldsymbol{\theta}'$:

$$\boldsymbol{\theta}' = \left\{ \begin{array}{c} \theta_1' \\ \theta_2' \end{array} \right\} = \left\{ \begin{array}{c} \theta_2 \\ -\frac{g}{L}\theta_1 \end{array} \right\}$$

The second-order Runge-Kutta method is defined as

$$\theta_{i+1} = \theta_i + \frac{h}{2}[k_1 + k_2]$$

$$k_1 = \boldsymbol{f}(t^{(i)}, \boldsymbol{\theta}^{(i)}) = \begin{pmatrix} \theta_2^{(i)} \\ -g\theta_1^{(i)}/L \end{pmatrix}$$

$$k_2 = \boldsymbol{f}(t^{(i)} + h, \boldsymbol{\theta}^{(i)} + hk_1) = \begin{pmatrix} \theta_1^{(i)} + h\theta_2^{(i)} \\ \theta_2^{(i)} - hg\theta_1^{(i)}/L \end{pmatrix}$$

$$\boldsymbol{\theta}^{(i+1)} = \boldsymbol{\theta}^{(i)} + \frac{h}{2}[k_1 + k_2] = \begin{pmatrix} \theta_1^{(i)} + 0.5h\left(2\theta_2^{(i)} - hg\theta_1^{(i)}/L\right) \\ \theta_2^{(i)} - 0.5gh/L\left(2\theta_1^{(i)} + h\theta_2^{(i)}\right) \end{pmatrix}$$

For 2 steps we have that $h = -0.5\ s$, while for 4 steps, $h = -0.25\ s$. The initial condition is $\theta_0 = 0.4\ rad$.
For 2 steps, $i = 0$ $\left(\theta_1^{(0)} = 0.4,\ \theta_2^{(0)} = 0\right)$:

$$\boldsymbol{\theta}^{(1)} = \boldsymbol{\theta}^{(0)} + \frac{h}{2}[k_1 + k_2] = \begin{pmatrix} 0.4 - 0.25\,(2\cdot 0 + 0.5\cdot 9.8\cdot 0.4) \\ 0 + 0.25\cdot 9.8\,(2\cdot 0.4 - 0.5\cdot 0) \end{pmatrix} = \begin{pmatrix} -0.09 \\ 1.96 \end{pmatrix}$$

For $i = 1$ $\left(\theta_1^{(1)} = -0.09,\ \theta_2^{(1)} = 1.96\right)$:

$$\boldsymbol{\theta}^{(2)} = \boldsymbol{\theta}^{(1)} + \frac{h}{2}[k_1 + k_2] = \begin{pmatrix} -0.09 - 0.25\,(2\cdot 1.96 - 0.5\cdot 9.8\cdot 0.09) \\ 1.96 - 0.25\cdot 9.8\,(2\cdot 0.09 + 0.5\cdot 1.96) \end{pmatrix} = \begin{pmatrix} -0.95975 \\ -0.882 \end{pmatrix}$$

Doing the same now for 4 steps. Starting with $i = 0$ $\left(\theta_1^{(0)} = 0.4,\ \theta_2^{(0)} = 0\right)$:

$$\boldsymbol{\theta}^{(1)} = \boldsymbol{\theta}^{(0)} + \frac{h}{2}[k_1 + k_2] = \begin{pmatrix} 0.4 - 0.125\,(2\cdot 0 + 0.25\cdot 9.8\cdot 0.4) \\ 0 + 0.125\cdot 9.8\,(2\cdot 0.4 - 0.25\cdot 0) \end{pmatrix} = \begin{pmatrix} 0.2775 \\ 0.98 \end{pmatrix}$$

For $i = 1$ $\left(\theta_1^{(1)} = 0.2775,\ \theta_2^{(1)} = 0.98\right)$:

$$\boldsymbol{\theta}^{(2)} = \boldsymbol{\theta}^{(1)} + \frac{h}{2}[k_1 + k_2] = \begin{pmatrix} 0.2775 - 0.125\,(2\cdot 0.98 + 0.25\cdot 9.8\cdot 0.2775) \\ 0.98 + 0.125\cdot 9.8\,(2\cdot 0.2775 - 0.25\cdot 0.98) \end{pmatrix} = \begin{pmatrix} -0.052484 \\ 1.35975 \end{pmatrix}$$

For $i = 2$ $\left(\theta_1^{(1)} = -0.052484,\ \theta_2^{(1)} = 1.35975\right)$:

$$\boldsymbol{\theta}^{(3)} = \boldsymbol{\theta}^{(2)} + \frac{h}{2}[k_1 + k_2] = \begin{pmatrix} -0.052484 - 0.125\,(2\cdot 1.35975 - 0.25\cdot 9.8\cdot 0.052484) \\ 1.35975 - 0.125\cdot 9.8\,(2\cdot 0.052484 + 0.25\cdot 1.35975) \end{pmatrix} = \begin{pmatrix} -0.376348 \\ 0.814741 \end{pmatrix}$$

For $i = 3$ $\left(\theta_1^{(1)} = -0.376348,\ \theta_2^{(1)} = 0.814741\right)$:

$$\boldsymbol{\theta}^{(4)} = \boldsymbol{\theta}^{(3)} + \frac{h}{2}[k_1 + k_2] = \begin{pmatrix} -0.376348 - 0.125\,(2 \cdot 0.814741 - 0.25 \cdot 9.8 \cdot 0.376348) \\ 0.814741 - 0.125 \cdot 9.8\,(2 \cdot 0.376348 + 0.25 \cdot 0.814741) \end{pmatrix} = \begin{pmatrix} -0.464777 \\ -0.356826 \end{pmatrix}$$

For 2 steps, the final position at $t = 0$ is $\theta = -0.95975$, while for 4 steps the final position is $\theta = -0.464777$. From the analytical solution, $\theta(t) = 0.4\cos(\sqrt{g/L}(t-1))$, we get that the exact solution is $\theta = -0.4$.

b) Using the approximations obtained in a), compute an approximation of the relative error in the solution computed with 2 steps.

The relative error is computed as

$$\epsilon = \left|\frac{\theta - \theta_2^{(2)}}{\theta}\right|$$

with $\theta = -0.4\ rad$ and $\theta_2^{(2)} = -0.95975\ rad$. Replacing values

$$\epsilon = \left|\frac{-0.4 + 0.95975}{-0.4}\right| = 1.399375 = 139.9375\%$$

For reference, using $\theta_2^{(4)} = -0.464777\ rad$:

$$\epsilon = \left|\frac{-0.4 + 0.464777}{-0.4}\right| = 0.1619425 = 16.19425\%$$

which is an almost 9 times reduction of the error.

c) Propose a time step h to obtain an approximation with a relative error three orders of magnitude smaller.

We already have a result with a relative error in the 10^{-1} order of magnitude with $h = 0.25$, and we are targeting an error of the order 10^{-4}. Since doubling the size of steps has reduced the order of magnitude of the error by one, it's only logical to propose doubling 3 times the number of steps to obtain an error in the desired order. Since we have 2^2 steps, $n = 2^5$ is tried with a MATLAB script, which gives

$$\theta_2^{(32)} = -0.400139080486129$$

for which the relative error is

$$\epsilon = \left|\frac{-0.4 + 0.400139080486129}{-0.4}\right| = 3.477012153234316 \cdot 10^{-4} = 0.034770121532343\%$$

which is the error with the desired order of magnitude.

2. Consider the initial value problem

$$\frac{dy}{dx} = y - x^2 + 1 \qquad x \in (0,1)$$
$$y(0) = 1$$

a) Solve the initial value problem using the Euler method with step $h = 0.25$.

The Euler method is defined in every step as

$$Y_{i+1} = Y_i + hf(x_i, y_i)$$

We have 4 steps:

$$y_{(1)} = y_{(0)} + hf(x_{(0)}, y_{(0)}) = 1 + 0.25 \cdot (1 - 0^2 + 1) = 1.5$$
$$y_{(2)} = y_{(1)} + hf(x_{(1)}, y_{(1)}) = 1.5 + 0.25 \cdot (1.5 - 0.25^2 + 1) = 2.109375$$
$$y_{(3)} = y_{(2)} + hf(x_{(2)}, y_{(2)}) = 2.109375 + 0.25 \cdot (2.109375 - 0.5^2 + 1) = 2.824219$$
$$y_{(4)} = y_{(3)} + hf(x_{(3)}, y_{(3)}) = 2.824219 + 0.25 \cdot (2.824219 - 0.75^2 + 1) = 3.639649$$

b) Compute the solution using the Heun method with a step h such that the computational cost is equivalent to the computational cost in a).

Euler method takes one computation per step, while Heun method takes two computations per step (the computation on k's and the next step). So, we use Heun method with a step double the size than Euler, $h = 0.5$. Heun method is written as

$$y_{i+1} = y_i + \frac{1}{2}[k_1 + k_2]$$
$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + h, y_i + hk_1)$$

for $i = 0$:

$$k_1 = 1 - 0^2 + 1 = 2$$
$$k_2 = (1 + 0.5 \cdot 2) - 0.5^2 + 1 = 3.75$$
$$y_1 = 1 + 0.5 \cdot [2 + 3.75] = 3.875$$

for $i = 1$:

$$k_1 = 3.875 - 0.5^2 + 1 = 4.625$$
$$k_2 = (3.875 + 0.5 \cdot 4.625) - 1^2 + 1 = 6.1875$$
$$y_2 = 3.875 + 0.5 \cdot [4.625 + 6.1875] = 9.28125$$

3. The ordinary differential equation

$$\frac{dy}{dx} = f(x, y)$$

is defined over the domain $(0, 1)$, and is to be solved numerically subject to the initial condition $y(0) = 1$, where $y(x)$ is the exact solution. The forward Euler method for integrating the above differential equation is written as

$$Y_{i+1} = Y_i + hf(x_i, Y_i)$$

where Y_i denotes the discrete solution at node i, with position x_i, of a uniform grid of nodes of constant grid interval size h and $x_{i+1} = x_i + h$.

a) Using a Taylor series expansion, deduce the leading truncation error of the scheme. Is the method consistent? Explain your answer.

Let $y(x_i)$ be the solution of the ODE at the point x_i, and $y(x_{i+1}) = y(x_i + h)$ the solution at the next point. The Taylor series around $x = a$ is defined as

$$y(x) = y(a) + y'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}y'''(a)(x-a)^3 + \cdots$$

So taking the variable $x \equiv x_i + h = x_{i+1}$ and $a \equiv x_i$, we obtain

$$\begin{aligned} y(x_i + h) &= y(x_i) + y'(x_i)(x_i + h - x_i) + \mathcal{O}((x_i + h - x_i)^2) \\ y(x_{i+1}) &= y(x_i) + hy'(x_i) + \mathcal{O}(h^2) \end{aligned}$$

Since $y'(x) = f(x, y)$, we finally obtain the expression

$$\begin{aligned} y(x_{i+1}) &= y(x_i) + hf(x_i, y(x_i)) \\ y_{i+1} &= y_i + hf(x_i, y_i) \end{aligned}$$

b) State the backward Euler method for integrating the above differential equation where $f(x, y)$ is a general non-linear function of x and y.

For the Backward Euler method we apply the same procedure as before, with the difference that $x \equiv x_{i+1} - h = x_i$ and $a = x_{i+1}$. Doing the Taylor expansion we obtain

$$\begin{aligned} y(x_{i+1} - h) &= y(x_{i+1}) + y'(x_{i+1})(x_{i+1} - h - x_{i+1}) + \mathcal{O}((x_{i+1} - h - x_{i+1})^2) \\ y(x_i) &= y(x_{i+1}) - hy'(x_{i+1}) + \mathcal{O}(h^2) \\ y(x_i) &= y(x_{i+1}) - hf(x_{i+1}, y(x_{i+1})) \\ y_i &= y_{i+1} - hf(x_{i+1}, y_{i+1}) \\ y_{i+1} &= y_i + hf(x_{i+1}, y_{i+1}) \end{aligned}$$

Which is implicit since y_{i+1} is dependant on itself.

c) Deduce the stability limits of the respective forward Euler method and backward Euler method for the model equation $dy/dx = -\lambda y$ where λ is a positive real constant.

The analytical solution for the problem is

$$y(x) = y_0 \mathrm{e}^{-\lambda x}$$

Stability demands that the solution tends to zero at large x. This implies, given the condition for λ, that the analytical problem converges. Using the Forward Euler method:

$$\begin{aligned} y_{i+1} &= y_i + hf(x_i, y_i) \\ y_{i+1} &= y_i - h\lambda y_i \\ y_{i+1} &= (1 - h\lambda)y_i \end{aligned}$$

By induction, we get that

$$y_k = y_0(1 - h\lambda)^k$$

and this should also tend to zero, which is true when $|1-h\lambda|$ is less than one. Written as an inequality

$$|1 - h\lambda| < 1$$

$$-1 < 1 - h\lambda < 1$$

$$0 < h\lambda < 2$$

So, for the condition set for λ, we need that $h\lambda$ has a value in the interval $]0, 2[$. Now for the Backward Euler method:

$$\begin{aligned} y_{i+1} &= y_i + hf(x_{i+1}, y_{i+1}) \\ y_{i+1} &= y_i - h\lambda y_{i+1} \\ y_i &= (1 + h\lambda)y_{i+1} \end{aligned}$$

By induction, we get that

$$y_k = y_0 \left(\frac{1}{1 + h\lambda}\right)^k$$

and we get a similar convergence condition:

$$\left|\frac{1}{1 + h\lambda}\right| < 1$$

$$|1 + h\lambda| > 1$$

this gives two cases:

$$\begin{aligned} 1 + h\lambda &> 1 \\ 1 + h\lambda &< -1 \end{aligned}$$

so finally the condition is

$$\begin{aligned} h\lambda &> 0 \\ h\lambda &< -2 \end{aligned}$$

which is a bigger range than the one covered by the Forward Euler Method.

d) Use the backward Euler method to compute the numerical solution of the ordinary differential equation

$$\frac{dy}{dx} = -25y^{3.5}$$

with initial condition $y(0) = 1$, by hand for two steps with grid interval size $h = 1/10$ (Use 2 Newton iterations per step for this calculation).

For Backward Euler:

$$\begin{aligned} y_{i+1} &= y_i + hf(x_{i+1}, y_{i+1}) \\ y_{i+1} &= y_i - h(25y_{i+1}^{3.5}) \end{aligned}$$

For $h = 1/10$ we have

$$\begin{aligned} y_{i+1} &= y_i - 2.5y_{i+1}^{3.5} \\ f(y_{i+1}) &= y_i - y_{i+1} - 2.5y_{i+1}^{3.5} \\ f'(y_{i+1}) &= -1 - 8.75y_{i+1}^{2.5} \end{aligned}$$

And for 2 steps, $i = 0$:

$$y_{(1)} = 1 - 2.5y_{(1)}^{3.5} \quad \longrightarrow \quad \begin{cases} f(y_{(1)}) = 1 - y_{(1)} - 2.5y_{(1)}^{3.5} \\ f'(y_{(1)}) = -1 - 8.75y_{(1)}^{2.5} \end{cases}$$

$y_{(1)}^{0} = 0.6$ (Obtained by inspection)

$$y_{(1)}^{1} = 0.6 - \frac{1 - 0.6 - 2.5 \cdot 0.6^{3.5}}{-1 - 8.75 \cdot 0.6^{2.5}} = 0.594685$$

$$y_{(1)}^{2} = 0.594685 - \frac{1 - 0.594685 - 2.5 \cdot 0.594685^{3.5}}{-1 - 8.75 \cdot 0.594685^{2.5}} = 0.594643 \quad \longrightarrow \quad f(y_{(1)}^{2}) = -8.937383 \cdot 10^{-9}$$

$y_{(1)} = 0.594643$

For $i = 1$:

$$y_{(2)} = 0.594643 - 2.5y_{(1)}^{3.5} \quad \longrightarrow \quad \begin{cases} f(y_{(2)}) = 0.594643 - y_{(2)} - 2.5y_{(2)}^{3.5} \\ f'(y_{(2)}) = -1 - 8.75y_{(2)}^{2.5} \end{cases}$$

$y_{(2)}^{0} = 0.45$ (Obtained by inspection)

$$y_{(2)}^{1} = 0.45 - \frac{0.594643 - 0.45 - 2.5 \cdot 0.45^{3.5}}{-1 - 8.75 \cdot 0.45^{2.5}} = 0.446263$$

$$y_{(2)}^{2} = 0.446263 - \frac{0.594643 - 0.446263 - 2.5 \cdot 0.446263^{3.5}}{-1 - 8.75 \cdot 0.446263^{2.5}} = 0.446242 \quad \longrightarrow \quad f(y_{(2)}^{2}) = 9.121735 \cdot 10^{-9}$$

$y_{(2)} = 0.446242$

From the analytical solution shown in part f, we obtain that $y(0.1) = 0.452757$ and $y(0.1) = 0.353075$.

e) Use the forward Euler method to compute the numerical solution of the above ordinary differential equation with same initial condition by hand for two steps with grid interval size $h = 1/10$.

We have 2 steps:

$$\begin{aligned} y_{i+1} &= y_i + hf(x_i, y_i) \\ y_{i+1} &= y_i - h(25y_i^{3.5}) \\ y_{(1)} &= y_{(0)} - h(25y_{(0)}^{3.5}) = 1 - 0.1 \cdot (25 \cdot 1^{3.5}) = -1.5 \\ y_{(2)} &= y_{(1)} - h(25y_{(1)}^{3.5}) = -1.5 - 0.1 \cdot (25 \cdot (-1.5)^{3.5}) = -1.5 + 10.333785i \end{aligned}$$

As can be seen, the step is to big to proceed, giving complex numbers as a result.

f) The analytical solution is

$$y(x) = \left(\frac{125x + 2}{2}\right)^{-2/5}$$

Using Matlab codes, indicate the maximum stable interval size possible for forward Euler method from the following; $h = 1/10$, $h = 1/15$, $h = 1/30$, $h = 1/45$, $h = 1/90$. How does your choice compare with the stability condition?

Using MATLAB, the solutions obtained with the proposed h values are

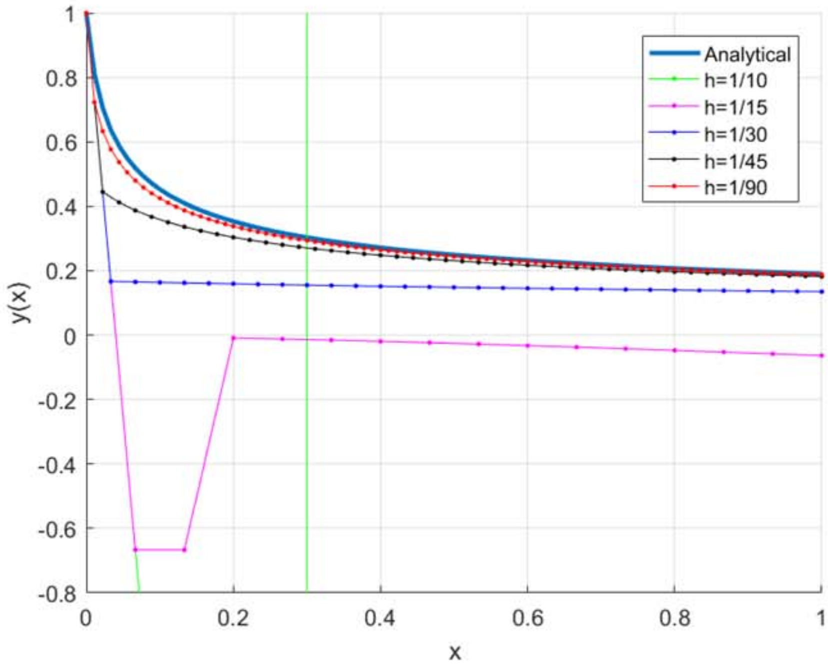

Figure 1: integration for different step sizes

As can be seen, the first to step sizes are unstable, giving complex numbers as results. From steps 1/30 and less the results converge and improve with more points used for the integration. From the stability conditions derived from before, we have $\lambda = 25$, so we should expect stability for $h < 1/12.5$. In this case, stability is not corresponding with this condition, probably considering the fact that $f(x, y)$ is not a linear function as defined in part c. In this particular case, stability is achieved from $h = 1/25$ and smaller.