Arzu Ahmadova

21. / 11 / 16.

Exercises (Basics)

Solution:

$$x^0 = \sqrt[3]{20} = 2{,}7144 17617$$

$\boxed{1^{st}}$

$$f(x) := x^3 + 2x^2 + 10x - 20 = 0 \qquad (1)$$

$$f'(x) = 3x^2 + 4x + 10, \quad \forall x.$$

It is clear that, $f'(x) \neq 0$, $\forall x \in \mathbb{R}$

Newton's method takes on the more familiar mathematical form: $u^{k+1} = u^k - \frac{f(u^k)}{f'(u^k)}$, $k = 0, 1, \ldots$

So we can compute (1) by the method:

$$x^1 = x^0 - \frac{f(x^0)}{f'(x^0)} = \sqrt[3]{20} - \frac{f(\sqrt[3]{20})}{f'(\sqrt[3]{20})} = 1{,}73959.$$

$$x^2 = x^1 - \frac{f(x^1)}{f'(x^1)} = 1{,}73959 - \frac{f(1{,}73959)}{f'(1{,}73959)} = 1{,}40497$$

$$x^3 = x^2 - \frac{f(x^2)}{f'(x^2)} = 1.40497 - \frac{f(1{,}40497)}{f'(1{,}40497)} = 1{,}36918$$

$$x^4 = x^3 - \frac{f(x^3)}{f'(x^3)} = 1.36881$$

$$x^5 = x^4 - \frac{f(x^4)}{f'(x^4)} = 1.36881$$

Apply analytical solution: α - is a analytical solution of (1): $\alpha = 1.36881$.

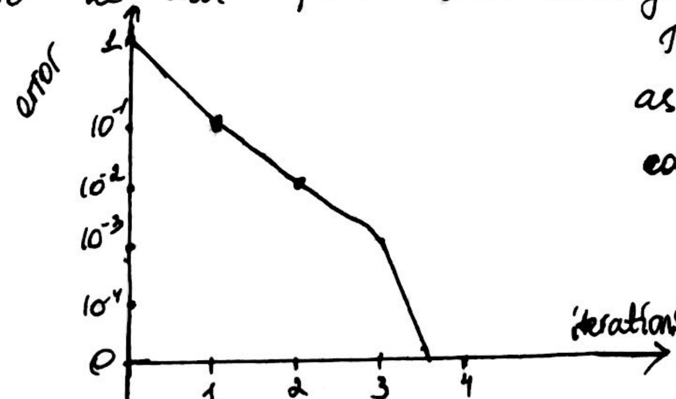
Compute the error r for each iteration:

$$r_0 = \left|\frac{x^0 - \alpha}{\alpha}\right| = \left|\frac{2{,}71441761 7 - 1.36881}{1.36881}\right| = 0.98305;$$

$$r_1 = \left|\frac{x^1 - \alpha}{\alpha}\right| = 0{,}27088; \quad r_2 = \left|\frac{x^2 - \alpha}{\alpha}\right| = 0{,}02642$$

$$r_3 = \left|\frac{x^3 - \alpha}{\alpha}\right| = 2.7031 \cdot 10^{-4}; \quad r_4 = \left|\frac{x^4 - \alpha}{\alpha}\right| = 0.$$

So we can plot the convergence graphic:



The method behaves as expected, with quadratic convergence as can be observed in the plot and also in the evalution of the errors (x^4 - last iterations

5d) a) The minimum number of integration points corresponds to a Gauss quadrature: Order $2n+1$ for $n+1$ integration points. Thus $n=1$ ·(2 Gauss integration points) leads to order 3.
The integration points in $[-1;1]$ are: ⇒
[So we have to change of variable from $[-1;1]$ to $[0,1]$]

$$\Rightarrow z_0 = \frac{\sqrt{3}}{3} \; ; \; w_0 = 1$$
$$z_1 = -\frac{\sqrt{3}}{3} \; ; \; w_1 = 1$$

Applying the change of variable: (2)

$$x = \frac{a+b}{2} + \frac{b-a}{2} z.$$

$$x = \frac{1}{2} + \frac{1}{2} z \quad , \quad dx = \frac{1}{2} dz$$

$$I = \int_0^1 F(x)dx = \frac{1}{2}\int_{-1}^{1} F\left(\frac{1}{2}+\frac{1}{2}z\right)dz = \frac{1}{2}\int_{-1}^{1} f(z)dz =$$
$$= \frac{1}{2}\sum_{i=0}^{1} w_i f(z_i) = \frac{1}{2}\cdot\sum_{i=0}^{1} w_i F\left(\frac{1}{2}z_i + \frac{1}{2}\right) = \frac{1}{2}\sum_{i=0}^{1} w_i F(x_i)$$

with $a=0$, $b=1$, the points and weights for the interval $[0,1]$ are.

$$x_0 = \frac{1}{2} + \frac{1}{2}\cdot\frac{\sqrt{3}}{3} = \frac{1}{2} + \frac{\sqrt{3}}{6} \quad , \quad w_0 = \frac{1}{2}.$$
$$x_1 = \frac{1}{2} - \frac{1}{2}\cdot\frac{\sqrt{3}}{3} = \frac{1}{2} - \frac{\sqrt{3}}{6.} \quad ; \quad w_1 = \frac{1}{2}$$

b) we have to consider that the integration points are given, weights can be computed. We use Newton-Cotes quadratures, this quadrature ~~with~~ is 3 points, and also third-order quadrature; weights corresponds to an open Newton-Cotes quadrature. with 3 points

$$x_s = \left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1\right].$$
$$w = [w_0, w_1, w_2, w_3]$$

We can write open Simpsons quadrature;

$$I = \int_0^1 F(x)dx = \frac{4h}{3}\left\{2F\left(\tfrac{1}{4}\right) - f\left(\tfrac{1}{2}\right) + 2f\left(\tfrac{3}{4}\right)\right\} + \cancel{\frac{14h^5}{45}} f'''(\mu) \; , \; h = \frac{1}{4}$$

$\to 0$ $\quad \mu \in [0,1)$

~~$w_0 = 0$~~

Solving the linear system of equation corresponding to imposing exact integration for $1, x, x^2, x^3$;

$$\int_0^1 1.dx = w_0 + w_1 + w_2 + w_3$$

$$\int_0^1 x\,dx = x_0 w_0 + x_1 w_1 + x_2 w_2 + x_3 w_3$$

The weights are 4 parameters to be determined imposing 4 conditions → exact integration for polynomials of degree less or equal to 3. In fact it is possible to define a third-order quadrature with any set of 4 points.

$$\begin{cases} 1 = \int_0^1 dx = w_0 + w_1 + w_2 + w_3 \\ \frac{1}{2} = \int_0^1 x\,dx = \frac{1}{4} w_0 + \frac{1}{2} w_1 + \frac{3}{4} w_2 + w_3 \\ \frac{1}{3} = \int_0^1 x^2 dx = \frac{1}{16} w_0 + \frac{1}{4} w_1 + \frac{9}{16} w_2 + w_3 \\ \frac{1}{4} = \int_0^1 x^3 dx = \frac{1}{64} w_0 + \frac{1}{8} w_1 + \frac{27}{64} w_2 + w_3 . \end{cases}$$

Solving these equation's system, we obtain:

$$w_0 = \frac{2}{3},\ w_1 = -\frac{1}{3},\ w_2 = \frac{2}{3},\ w_3 = 0.$$

$\boxed{6^{th}}$ Solution of Gauss quadrature with $n+1$ points achieves order $2n+1$, that is, it integrates exactly polynomials of degree less or equal to $2n+1$.

b) i) $\int_0^1 \sin x\,dx$ → NO!, this integrated function is not a polynomial, so we can not say that it will be integrated.

ii) $\int_0^1 x^3 dx$ → Yes!

according to explain of a) which $n+1 = 3$ integration points, polynomials degree $2n+1 = 5$ so we can integrate exactly

iii) $\int_0^1 x^4 dx$ → Yes! ~~so~~ it is clear that $4 \le 5$, so we can integrate exactly

iv) $\int_0^1 x^{5,5} dx$ → NO!

~~so~~ we can not guarantee that it will be integrated exactly, so, $5,5 > 5$.

7th

Solution: The first case is the integral of linear function. thus, both trapezoidal and Simpson's rules give the exact solution. In fact, the exact result is also obtained 4 using only one integration interval;

$\int_0^1 12x\,dx$ → ✓ trapezoidal rule. $f(x) = 12x$

$I = \int_0^1 12x\,dx = \frac{1}{4}\left(f(0) + 2f\left(\frac{1}{2}\right) + f(1) \right) =$ $h = \frac{1}{2}$

$= \frac{1}{4}(0 + 2.6 + 12] = 6 \Rightarrow I = 6$

✓ Simpson's rule

$I = \int_0^1 12x\,dx = \frac{1}{12}\left(f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1) \right)$

$= \frac{1}{12}\{0 + 12 + 12 + 36 + 12) = 6.$

In this case, both methods have integrated it exactly, and analytical solution is 6

$\int_0^1 (5x^3 + 2x)\,dx$. ; 1) Trapezoidal rule:

$I = \int_0^1 (5x^3 + 2x)\,dx = \frac{1}{4}\left(f(0) + 2f\left(\frac{1}{2}\right) + f(1) \right\} =$

$= \frac{1}{4}\left(2\cdot\left[5\cdot\frac{1}{8} + 1\right] + 7 \right) = \frac{1}{4}\left[\frac{13}{4} + 7 \right) = \frac{41}{16}$

2) Simpson's rule:

$I = \int_0^1 (5x^3 + 2x)\,dx = \frac{1}{12}\left(f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1) \right\} = \frac{1}{12}\cdot\left(\frac{37}{16} + \frac{13}{4} + \frac{231}{16} + 7 \right) = \frac{9}{4}$

In this case, analytical solution is $\frac{9}{4}$;
we have an error: $E = \left| I - \frac{9}{4} \right| = \frac{5}{16}$

- Both methods behaved as expected. The second one is a function of a third degree polynomial. Using trapezoidal rule we can only approximate the integral whereas Simpson's rule yields to the exact value (because it is a third-order quadrature)

$\boxed{10^*}$ We can separete the integral in every direction dx and dy ; so.

$$I = \int_0^1 \int_0^1 \underbrace{(9x^3 + 8x^2)}_{I_1} \underbrace{(y^3 + y)}_{I_2} dx\,dy = \int_0^1 (9x^3 + 8x^2)\,dx \int_0^1 (y^3 + y)\,dy$$

$h = \frac{1}{2}$

$\frac{h}{3} = \frac{1}{6}$

Using Simpson's rule and compute each Integral

$$I_1 = \int_0^1 (9x^3 + 8x^2)\,dx = \frac{1}{6}\left(f(0) + 4f(\tfrac{1}{2}) + f(1)\right) =$$

$$= \frac{1}{6}\left(\frac{25}{2} + 17\right) = \frac{59}{12}$$

$$I_2 = \int_0^1 (y^3 + y)\,dy = \frac{1}{6}\left(f(0) + 4f(\tfrac{1}{2}) + f(1)\right) = \frac{1}{6}\left(\frac{5}{2} + 2\right) = \frac{3}{4}$$

So the result.

$$I = I_1 \cdot I_2 = \frac{59}{12} \cdot \frac{3}{4} = \frac{59}{16}$$

In each function $[I_1\ \&\ I_2]$ is third-~~order~~ degree polynomial, so Simpson's rule will be integraded it exactly; Let's compute analytical solution

$$I = \int_0^1 (9x^3 + 8x^2)\,dx \cdot \int_0^1 (y^3 + y)\,dy = \left[9\frac{x^4}{4} + \frac{8x^3}{3}\right]\Big|_0^1 \cdot \left[\frac{1}{4}y^4 + \frac{y^2}{2}\right]_0^1$$

$$= \frac{59}{12} \cdot \frac{3}{4} = \frac{59}{16} \Rightarrow E = 0,$$ so we obtain that the method behaved as expected.