NM4PDEs - Exercises Basics

1*. In 1225, Leonardo of Pisa (also known as Fibonacci) was requested to solve a collection of mathematical problems in order to justify his fame and prestige in the court of Federico II. One of the proposed problems can be formulated as the solution of a third degree polynomial equation

$$f(x) := x^3 + 2x^2 + 10x - 20 = 0 \tag{1}$$

Note that the solution of cubic equations was a extremely difficult problem in the $13th$ century. Here iterative methods are considered for the solution of equation (1).

Compute the unique real root of (1) with 4 iterations of Newton's method with the initial approximation $x^0 = \sqrt[3]{20}$ (which is obtained neglecting the monomials with x and x^2 in front of the monomial with x^3). Plot the convergence graphic. Does Newton's method behave as expected?

Polynomial equation

$$f(x) = x^3 + 2x^2 + 10x - 20 = 0$$

Its derivative

$$f'(x) = 3x^2 + 4x + 10$$

Initial approx

$$x^0 = \sqrt[3]{20}$$

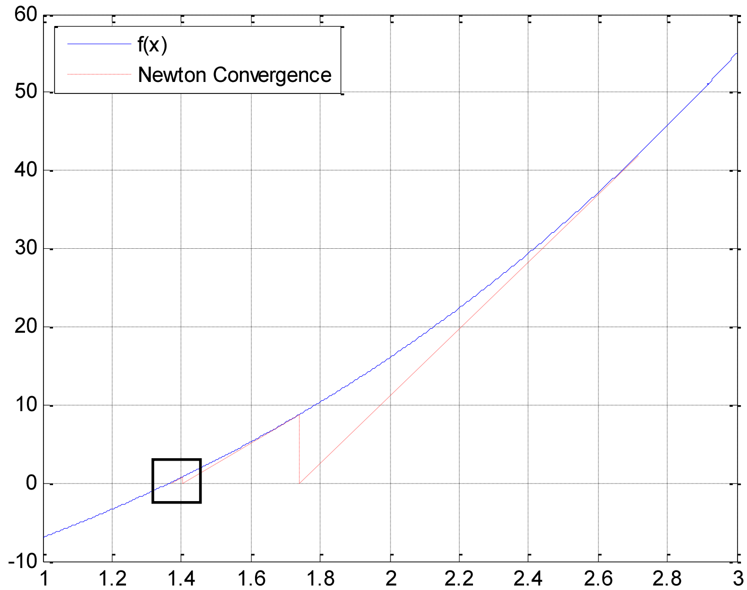
Successive solutions

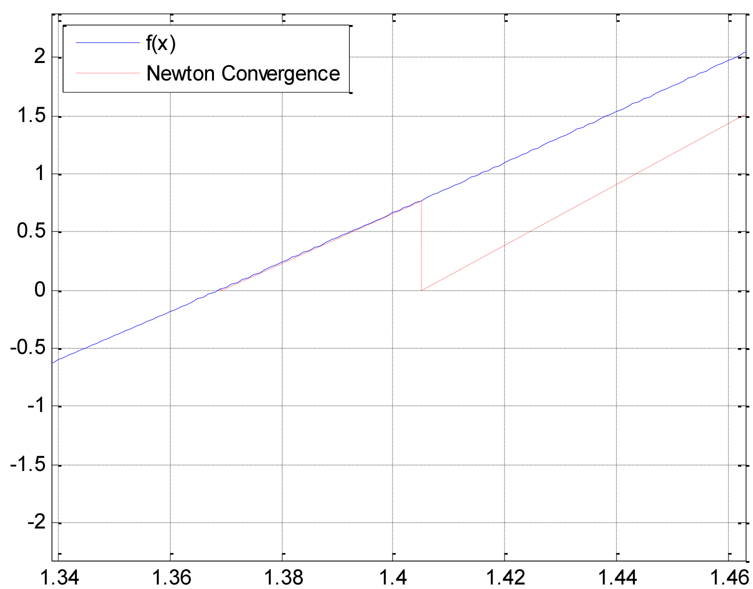
$$x^{k+1} = x^k + \Delta x^{k+1}$$

$$f(x^{k+1}) = f(x^k + \Delta x^{k+1}) \approx f(x^k) + f'(x^k)\Delta x^{k+1}$$

$$\Delta x^{k+1} = -\frac{f(x^k)}{f'(x^k)}$$

k	x^k	$f(x^k)$	$f'(x^k)$	Δx^{k+1}	x^{k+1}
0	$\sqrt[3]{20}$	41.8803	42.9619	-0.9748	1.7396
1	1.7396	8.7126	26.0369	-0.3346	1.4050
2	1.4050	0.7708	21.5417	-0.0358	1.3692
3	1.3692	0.0079	21.1007	-3.7498e-4	1.3688





The Newton method implemented for this problem behaves as expected as a quadratic convergence is observed according to evolution of Δx^{k+1}.

5*. We are interested in the definition of third-order numerical quadratures in interval $(0,1)$

a) Determine the minimum number of integration points, and specify the integration points and weights.

b) Is it possible to obtain a third-order quadrature with the following four integration points: $x_0 = 1/4$, $x_1 = 1/2$, $x_2 = 3/4$ and $x_3 = 1$? If it is possible, compute the corresponding weights; otherwise, justify why not.

Third order numerical quadrature

a) Gauss quadratures maximize the accuracy without increasing to total number of quadrature points. This is done by chosing the integration points at the same time that the weights are chosen. This way, higher order quadratures can be obtained. Integration points and weights are chosen so polynomials of degree up to $2n+1$ are integrated exactly.

$$I = \int_0^1 f(x) \; \mathrm{d}x = \sum_{i=0}^{n} w_i f(x_i) + E_n$$

$$E_n = \Omega_n f^{2n+2)}$$

Third-order quadratures can be achieved with $n = 1$ point.

b) In general it is not possible. Newton-Cotes uses equally spaced points, and there are enough integration points in the problem but the method requires the first and last points to be coincident with the interval bounds. Gauss is not possible either because integration points are degrees of freedom of the problem, do they can not be imposed or chosen beforehand.

6*. *a)* If $n+1$ points Gaussian quadrature is used for numerical integration state the order of the polynomial that is integrated exactly.

b) If $n = 2$ is selected for Gaussian quadrature, which (if any) of the following integrals will be integrated exactly?

i) $\int_0^1 \sin x \, dx$ *ii)* $\int_0^1 x^3 \, dx$ *iii)* $\int_0^1 x^4 \, dx$ *iv)* $\int_0^1 x^{5.5} \, dx$

i. Exact in tegration of polynomials of degree up to $2n+1$ is achieved with n points Gaussian quadrature. Therefore $2(n+1)+1 = 2n+3$ degree polynomials are integrated exactly with $n+1$ integration points.

ii. With $n = 2$ points polynomials of degree up to 5 are integrated exactly. Therefore only integrals ii and iii are solved exactly. These are lower order polynomials. Integral i may be decomposed in an infinity of polynomials, some of which will be of order higher than 5. Integral iv is not a polynomial with an integer exponent so it doesn't fit with the definition of polynomial.

7*. Compute $\int_0^1 12x\,dx$, $\int_0^1 (5x^3+2x)\,dx$ by hand calculation using

i) Trapezoidal rule over 2 uniform intervals

ii) Simpson's rule over 2 uniform intervals

Compute the error of both approximations. Are the methods behaving as expected?

i. Trapezoidal rule over 2 uniform intervals

Uniform intervals are $[0, \frac{1}{2}, 1]$, $h = \frac{1}{2}$.

First integral:

Being the derivative of $f(x) = 12x$, $f'(x) = 12$, $f''(x) = 0$

$$I = \frac{1}{4}\left(f(0) + f(\tfrac{1}{2})\right) \quad + \quad \frac{1}{4}\left(f(\tfrac{1}{2}) + f(1)\right)$$

$$I = \frac{1}{4}\left(0 + \frac{12}{2}\right) \quad + \quad \frac{1}{4}\left(\frac{12}{2} + 12\right) \quad = \quad 6$$

$$E_1 = -\frac{h^3}{12} f''(\mu) \quad = \quad 0$$

Second integral:

Being the derivatives of $f(x) = 5x^3 + 2x$, $f'(x) = 15x^2 + 2$, $f''(x) = 30x$

$$I = \frac{1}{4}\left(f(0) + f(\tfrac{1}{2})\right) \quad + \quad \frac{1}{4}\left(f(\tfrac{1}{2}) + f(1)\right) \quad + E_1$$

$$I \approx \frac{1}{4}\left(0 + 5\left(\frac{1}{2}\right)^3 + 2\frac{1}{2}\right) \quad + \quad \frac{1}{4}\left(5\left(\frac{1}{2}\right)^3 + 2\frac{1}{2} + 5 + 2\right) \quad = \quad \frac{41}{16}$$

$$E_1 \approx -\frac{h^3}{12} f''(\mu) \quad = \quad -\frac{1}{96} f''(\mu)$$

ii. Simpson's rule over 2 uniform intervals

Uniform intervals are $[0, \frac{1}{2}, 1]$, $h = \frac{1}{2}$.

First integral:

Being the derivative of $f(x) = 12x$, $f''(x) = 0$

$$I = \frac{1}{6}\left(f(0) + 4f(\tfrac{1}{2}) + f(1)\right) \quad + \quad E_2$$

$$I \approx \frac{1}{6}(0 + 4 \cdot 6 + 12) \quad = \quad 6$$

$$E_2 \approx -\frac{mh^5}{90} f^{4)}(\mu) = 0$$

Second integral:

Being the derivatives of $f(x) = 5x^3 + 2x$, $f''(x) = 15x^2 + 2$, $f'''(x) = 30x, f^{4)}(x) = 30$

$$I = \frac{1}{6}\big(f(0) + 4f(½) + f(1)\big) \quad + \quad E_2$$

$$I \approx \frac{1}{6}\left(0 + 4\left(5\left(\frac{1}{2}\right)^3 + 2\frac{1}{2}\right) + 5 + 2\right) = \frac{27}{12}$$

$$E_2 \approx -\frac{mh^5}{90} f^{4)}(\mu) \quad = \quad \frac{2(½)^5}{90} f^{4)}(\mu) \quad = \quad \frac{1}{1440} f^{4)}(\mu)$$

The behaviour of Trapezoid integration and Simpson's rule give resulsts as expected: for the linear function both yield exact results; for the cubic polynomial, the Simpson's rule higher order results in a higher accuracy and lower error.

10*. Perform the numerical integration of

$$\int_0^1 \int_0^1 (9x^3 + 8x^2)(y^3 + y)\, dx\, dy$$

using Simpson's rule in each direction. Is the approximation behaving as expected?

$$\int_0^1 \int_0^1 (9x^3 + 8x^2)(y^3 + y) \quad \mathrm{d}x\, \mathrm{d}y$$

$$\int_0^1 \int_0^1 f(x)g(y) \quad \mathrm{d}x\, \mathrm{d}y$$

$$I = \left(\frac{h}{3}\right)^2 \left(f(0) + 4f(½) + f(1)\right)\left(g(0) + 4g(½) + g(1)\right)$$

$$f(0) = 0$$

$$f(½) = 25/8$$

$$f(1) = 17$$

$$g(0) = 0$$

$$g(½) = \frac{5}{8}$$

$$g(1) = 2$$

Therefore,

$$I = \frac{59}{16}$$

Analytically,

$$\int_0^1 \int_0^1 (9x^3 + 8x^2)(y^3 + y) \quad \mathrm{d}x\, \mathrm{d}y$$

$$= \left(\frac{9}{4}x^4 + \frac{8}{3}x^3\right)_0^1 \left(\frac{1}{4}y^4 + \frac{1}{2}y^2\right)_0^1$$

$$= \left(\frac{9}{4} + \frac{8}{3}\right)\left(\frac{1}{4} + \frac{1}{2}\right) \quad = \quad \frac{59}{16}$$

The method is behaving as expected. The double integral is totally decoupled and therefore x and y direcetions can be treated separately. For each direction a third degree polynomial is integrated with Simpson's rule which result must be exact.