1)

Given function is,

$$f(x) := x^3 + 2x^2 + 10x - 20 = 0$$

$$f'(x) = 3x^2 + 4x + 10$$

Initial approximation is,

$$x_0 = \sqrt[3]{20}$$

Newton's method is,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

1st iteration,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$f(x_0) = (\sqrt[3]{20})^3 + 2(\sqrt[3]{20})^2 + 10\sqrt[3]{20} - 20$$

$$= 41.8803$$

$$f'(x_0) = 3(\sqrt[3]{20})^2 + 4(\sqrt[3]{20}) + 10 = 42.9619$$

$$x_1 = \sqrt[3]{20} - \frac{41.8803}{42.9619} = 1.7396$$

2nd iteration,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$f(x_1) = f(1.7396) = (1.7396)^3 + 2(1.7396)^2 + 10 \times 1.7396 - 20 = 8.7126$$

$$f'(x_1) = f'(1.7396) = 3(1.7396)^2 + 4 \times (1.7396) + 10 = 26.0370$$

$$x_2 = 1.7396 - \frac{8.7126}{26.0370} = 1.4050$$

3rd iteration,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$f(x_2) = 0.7711$$

$$f'(x_2) = 3 \times (1.4050)^2 + 4(1.4050) + 10$$

$$= 21.5421$$

$$x_3 = 1.4050 - \frac{0.7711}{21.5421}$$

$$= 1.3692$$

$4^{th}$ iteration,

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$f(x_3) = 8.372 \times 10^{-3}$$

$$f'(x_3) = 3 \times (1.3692)^2 + 4 \times (1.3692) + 10$$

$$= 21.1009$$

$$x_4 = 1.3692 - \frac{8.372 \times 10^{-3}}{21.1009}$$

$$x_4 = 1.3688$$

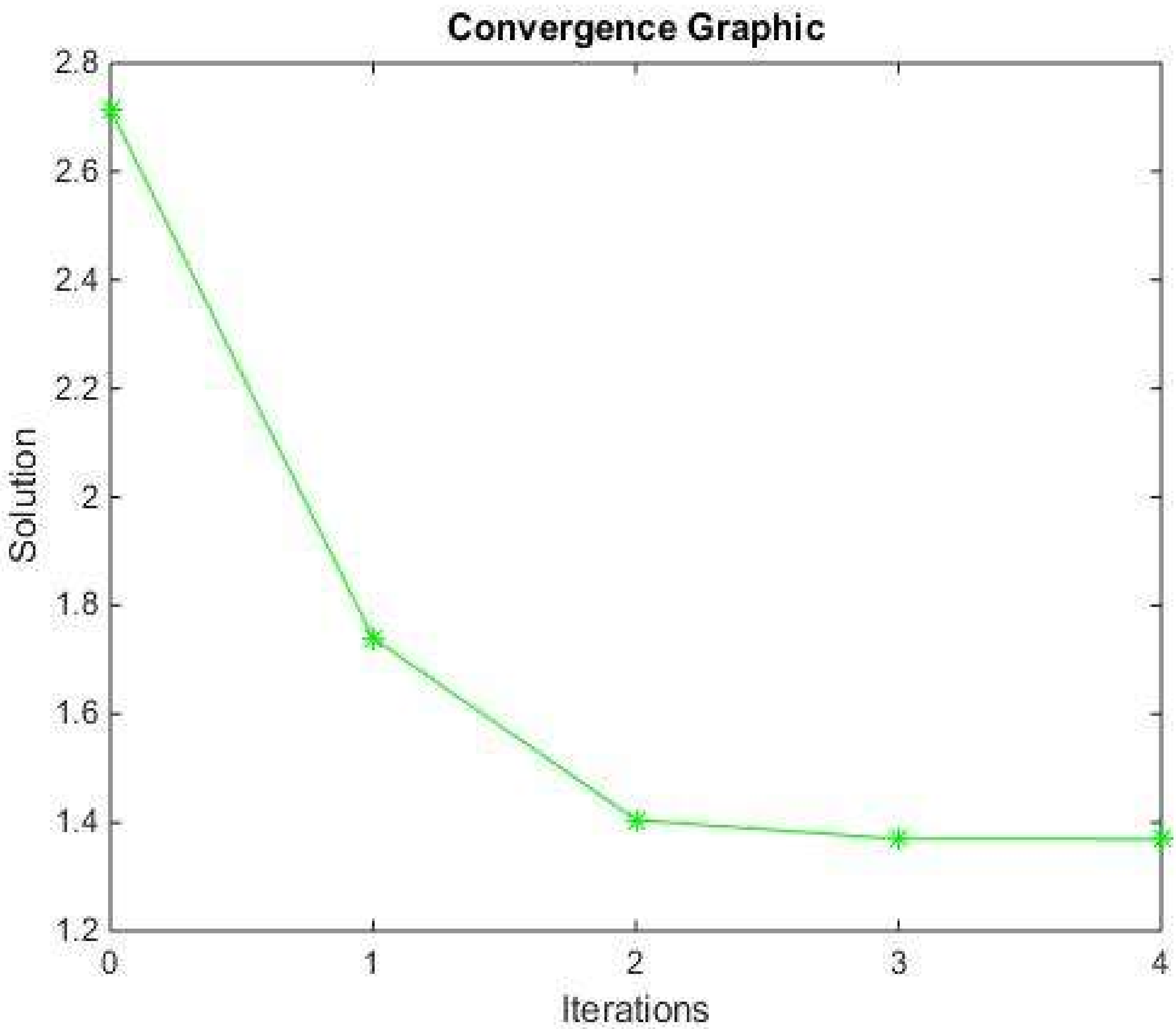
The unique real root of $f(x)$ is 1.3688

The convergence graph is shown in next page.

Yes, the newton's method behaves as expected.

The root converges to the actual root 1.3688

(error tends to zero).


Convergence Graphic
Solution
Iterations
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
0
1
2
3
4

5) Solution:-

a) We need third order numerical quadratures in interval (0,1).
Using Gauss-Legendre quadrature,

$2n+1 = 3$

$n = 1$

We have 1 interval and 2 integration points.

Hence, 2 integration points needed.

From the table for, (Gauss-Legendre table)

we have,

for, $\int_{-1}^{1} f(z)dz \approx \sum_{i=0}^{i=n} w_i f(z_i)$

$z_0 = -0.57735$ and $w_0 = 1$

$z_1 = 0.57735$ and $w_1 = 1$

transforming to required form,

$$\int_a^b F(x)\,dx \approx \frac{b-a}{2} \sum_{i=0}^{n} w_i F\left(\frac{(b-a)z_i + (b+a)}{2}\right)$$

$$\approx \frac{1-0}{2} \sum_{i=0}^{n} w_i F\left(\frac{(1-0)z_i + (1+0)}{2}\right)$$

$$\approx \sum_{i=0}^{n} \left(\frac{w_i}{2}\right) F\left(\frac{z_i+1}{2}\right)$$

the required weights are

$W_0 = \frac{w_0}{2} = 1/2$

$W_1 = \frac{w_1}{2} = 1/2$

and integer points are

$x_0 = \frac{z_0+1}{2} = \frac{-0.57735+1}{2}$

$x_0 \approx 0.21132$

$x_1 = \frac{z_1+1}{2} = \frac{0.57735+1}{2}$

$x_1 \approx 0.78868$

b) Consider, four points

$x_0 = 1/4,\ x_1 = 1/2,\ x_2 = 3/4,\ x = 1$

Let $f(x) = ax^3 + bx^2 + cx + d$

We have,

$$\int_0^1 f(x)\,dx = \sum_{i=0}^{i=4} w_i f(x_i)$$

$$\int_0^1 (ax^3 + bx^2 + cx + d)\,dx = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

$$\left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_0^1 = w_0 f(1/4) + w_1 f(1/2) + w_2 f(3/4) + w_3 f(1)$$

$$\Rightarrow \frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d = w_0\left[\frac{a}{64} + \frac{b}{16} + \frac{c}{4} + d\right] + w_1\left[\frac{a}{8} + \frac{b}{4} + \frac{c}{2} + d\right] + w_2\left[\frac{27a}{64} + \frac{9b}{16} + \frac{3c}{4} + d\right]$$
$$+ w_3\left[a + b + c + d\right]$$

Comparing co-efficient of a, b, c and d, we have

$$\frac{w_0}{64} + \frac{w_1}{8} + \frac{27w_2}{64} + w_3 = \frac{1}{4}$$

$$\frac{w_0}{16} + \frac{w_1}{4} + \frac{9w_2}{16} + w_3 = \frac{1}{3}$$

$$\frac{w_0}{4} + \frac{w_1}{2} + \frac{3w_2}{4} + w_3 = \frac{1}{2}$$

$$w_0 + w_1 + w_2 + w_3 = 1$$

Solving the above system of linear equations, we get the weights

$w_0 = 0.6667$

$w_1 = -0.3333$

$w_2 = 0.6667$

$w_3 = 0$

In open interval $(0,1)$ it is possible to obtain third-order quadrature with given integration points (since, $w_3 = 0$)

6) Solution

a) For $(n+1)$ points we have $n$ intervals.

For $n$ intervals in gaussian quadrature we can integrate exactly of polynomial of degree $\underline{\underline{2n+1}}$.

b) For $n=2$, (number of intervals)

$2n+1 = 2\times 2+1 = 5$.

The polynomials of degree less than or equal to 5 can be integrated exactly.

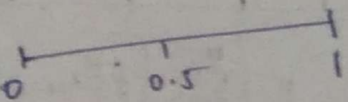
Hence, ii) $\int_0^1 x^3\,dx$ and iii) $\int_0^1 x^4\,dx$ can be integrated exactly.

7)

a) $\int_0^1 12x\,dx$

i) Using Trapezoidal rule over 2 uniform intervals.

| | 0 | 1 | 2 |
|---|---|---|---|
| $x$ | 0 | 0.5 | 1 |
| $f(x)$ | 0 | 6 | 12 |




$h = \frac{b-a}{2} = \frac{1}{2}$

$I = \sum_{i=1}^{2} \int f(x)\,dx =$

$I = \left(\sum_{i=0}^{1} w_i f_i\right)_{\text{over the intervals}} = I_1 + I_2$

$= \frac{h}{2}\left[f(x_0) + f(x_1)\right] + \frac{h}{2}\left[f(x_1) + f(x_2)\right]$

$= \frac{h}{2}\left[f(x_0) + 2f(x_1) + f(x_2)\right]$

$$= \frac{1/2}{2}\left[0 + 2\times 6 + 12\right]$$

$$= \frac{1}{4}[24]$$

$$= \underline{\underline{6}}$$

Error, $E_m^T = -\frac{(b-a)^3}{12m^2} f''(x)$

$$= \underline{\underline{0}} \qquad [\because f''=0]$$

The actual value of integral is,

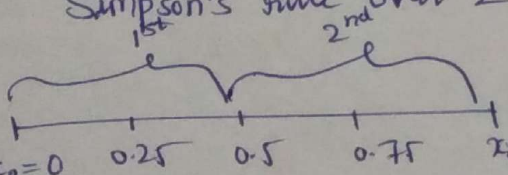
$$I = \int_0^1 12x\,dx = \left[12x^2/2\right]_0^1 = \underline{\underline{6}}$$

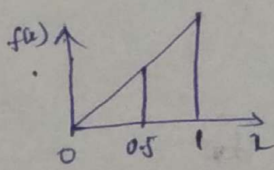
$$I_{actual} = I_{trapezoidal} + E_m^T$$

Hence, the method is behaving as expected

ii) $\int_0^1 12x\,dx$

Using Simpson's rule over 2 uniform intervals




| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $x$ | 0 | 0.25 | 0.5 | 0.75 | 1 |
| $f(x)$ | 0 | 3 | 6 | 9 | 12 |

$$h = \frac{b-a}{4} = 1/4$$

$$I = \frac{h}{3}\sum_{i=1}^{m}\left(f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})\right)$$

$$= \frac{h}{3}\left[f_0 + 4f_1 + f_2\right] + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4[f(x_1) + f(x_3)] + 2f(x_2) + f(x_4)\right]$$

$$= \frac{1/4}{3}\left[0 + 4[3+9] + 2\times 6 + 12\right]$$

$= \underline{\underline{\cdot 6}}$

$$E_m^s = -\frac{(b-a)^5}{2880m^4} \cdot f^4(\mu)$$

$= 0 \quad [\because f(\mu) = 0]$

$I_{actual} = I_{Simpson} + Error$

Hence, trapezoidal is behaving as expected.

i) $\int_0^1 (5x^3 + 2x)\, dx$ using trapezoidal rule over 2 intervals

$0 \quad 0.5 \quad 1$

$h = \frac{b-a}{n} = \frac{b-a}{2} = \frac{1}{2}$

| $x$ | 0 | 0.5 | 1 |
|---|---|---|---|
| $f(x)$ | 0 | 13/8 | 7 |

$$I = \frac{h}{2}[f(x_0) + f(x_1)] + \frac{h}{2}[f(x_1) + f(x_2)]$$

$$= \frac{h}{2}[f(x_0) + 2f(x_1) + f(x_2)]$$

$$= \frac{1/2}{2}[0 + 2 \times 13/8 + 7]$$

$$= \underline{\underline{2.5625}}$$

Error, $E_m^T = -\frac{(b-a)^3}{12m^2} f^2(\mu)$

$$= -\frac{1^3}{12 \times 4}[30\mu]$$

$$= \underline{\underline{\frac{-30\mu}{12 \times 4}}}$$

$0 \le \mu \le 1$

$\Rightarrow 0 \le \frac{-5\mu}{8} \le -5/8 \Rightarrow$ $E_m^T \le 0.3125 \times 2$

$E_m^T \le 0.625$

The actual value of integration,

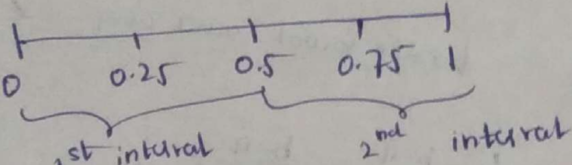
$$I = \int_0^1 (5x^3 + 2x)\,dx = \left[\frac{5x^4}{4} + \frac{2x^2}{2}\right]_0^1 = \left[\frac{5x^4}{4} + 1\right]_0^1 = \left[\frac{5}{4} + 1\right] = \underline{2.25}$$

$$I_{actual} - I_{trapizoidal} = 2.5625 - 2.25 = \underline{0.3125} = E_{actual}$$

$$\underline{E_{actual}} < (E^{T}_{rror})\ limit$$

Hence, the trapizoidal rule is behaving as expected.

ii) $\int_0^1 (5x^3 + 2x)\,dx$ over 2 intervals using simpson's rule,




| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $x$ | 0 | 0.25 | 0.5 | 0.75 | 1 |
| $f(x)$ | 0 | $\frac{37}{64}$ | $13/8$ | $231/64$ | 7 |

$$h = \frac{b-a}{4} = \underline{1/4}$$

$$I = I_1 + I_2$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] + \frac{h}{3}\left[f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4\left[f(x_1) + f(x_3)\right] + 2f(x_2) + f(x_4)\right]$$

$$= \frac{0.25}{3}\left[0 + 4\left[\frac{37}{64} + \frac{231}{64}\right) + 2\,(13/8) + 7\right]$$

$$= \underline{2.25}$$

$$E_m = -\frac{(b-a)^5}{2880m^4} f^4(w)$$

$$= 0 \quad \left[\because f^4 = 0\right]$$

$$I_{actual} = I_{sympson}$$

So, Simpson is behaving as expected.

10) Solution:-

$$\int_0^1\int_0^1 (9x^3+8x^2)(y^3+y)\,dx\,dy = \left[\int_0^1 (9x^3+8x^2)\,dx\right] \times \left[\int_0^1 (y^3+y)\,dy\right]$$

$$I = I_1 \times I_2$$

Calculate using simpson's rule.

$$I_1 = \int_0^1 (9x^3+8x^2)\,dx$$

0 0.5 1

$x_0$ $x_1$ $x_2$

$h = \frac{b-a}{n} = 1/2$

$$I_2 = \int_0^1 (y^3+y)\,dy$$

$h = \frac{b-a}{n} = 1/2$

$y_0$ (0) $y_1$ (0.5) $y_2$ (1)

| | 0 | 1 | 2 |
|---|---|---|---|
| $x$ | 0 | 0.5 | 1 |
| $f(x)$ | 0 | 25/8 | 17 |
| $g(y)$ | 0 | 5/8 | 2 |

$$I_1 = \sum_{i=0}^{2} w_i f_i$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$= \frac{1}{2\times3}\left[0 + 4\times\frac{25}{8} + 17\right]$$

$$= \frac{1}{6}\left[\frac{25}{2}+17\right]$$

$$I_1 = \frac{59}{12} = 4.9167$$

$$I_2 = \frac{h}{3}\left[f(y_0) + 4f(y_1) + f(y_2)\right]$$

$$= \frac{1}{2\times3}\left[0 + 5/8\times4 + 2\right]$$

$$= \frac{1}{6}\left[\frac{5}{2}+2\right]$$

$$= \frac{1}{6}\left[\frac{9}{2}\right]$$

$$I_2 = \frac{9}{12} = 0.75$$

$$I = I_1 \times I_2$$

$$= \frac{59}{12} \times \frac{9}{12}$$

$$I = 3.6875$$

$$E_T = E_1 + E_2$$

$$= \left[-\frac{(b-a)^5}{90n^5}\right] f^4(\eta) \times 2$$

$$= 0 \quad [\because f^4 = 0]$$

The actual value of I,

$$I = \int_0^1\int_0^1 (9x^3+8x^2)(y^3+y)\,dx\,dy$$

$$= \left[\frac{9x^4}{4}+\frac{8x^3}{3}\right]_0^1 \times \left[\frac{y^4}{4}+\frac{y^2}{2}\right]_0^1$$

$$= \left[\frac{9}{4}+\frac{8}{3}\right]\left[\frac{1}{4}+\frac{1}{2}\right]$$

$$= 3.6875$$

$\cancel{I_{simpson} = I_{actual}} + E.$

$$I_{actual} = I_{simpson} + E_T$$

Hence, simpson is behaving as expected.