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Numerical methods for partial differential equations
Individual HW-1
Basics

Question 1:

$$f_{(x)} = x^3 + 2x^2 + 10x - 20 = 0$$

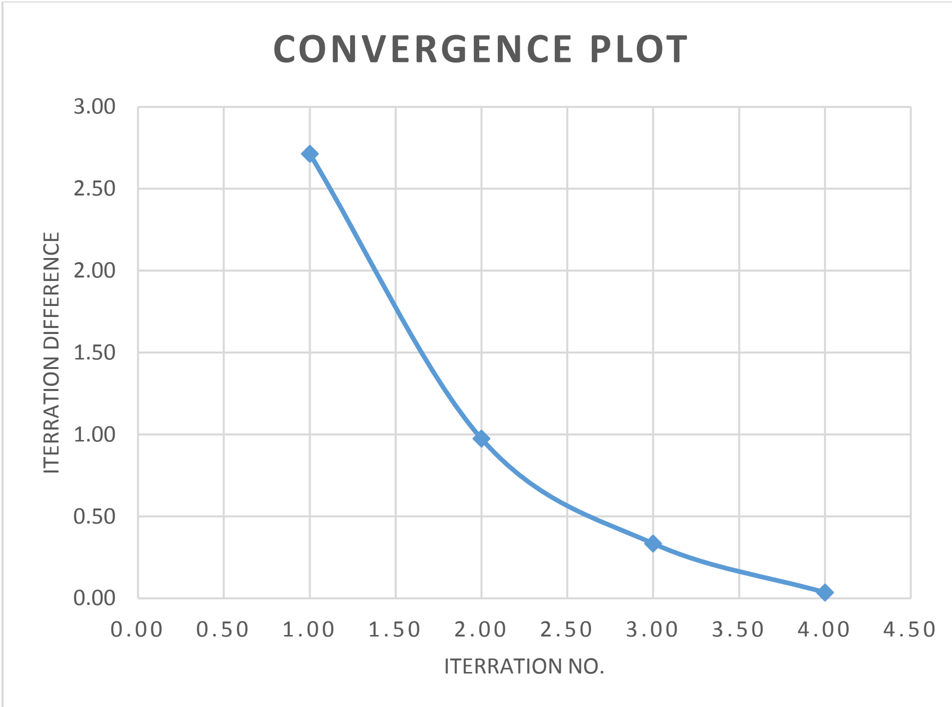
Compute the unique real root of (1) with 4 iterations of Newton's method with the initial approximation $X_0 = \sqrt[3]{20}$ (which is obtained neglecting the monomials with x and x_2 in front of the monomial with x_3) for the previous polynomial. Plot the convergence graphic. Does Newton's method behave as expected?

Answer

X0	f(X0)	f'(X0)	DX	X1
2.71	41.88	42.96	0.97	1.74
1.74	8.71	26.04	0.33	1.40
1.40	0.77	21.54	0.04	1.37
1.37	0.01	21.10	0.00	**1.37**

Iteration no.	Iteration difference
1	2.71
2	0.97
3	0.33
4	0.04

The unique real root of this polynomial is **1.37**.


CONVERGENCE PLOT
3.00
2.50
2.00
1.50
1.00
0.50
0.00
ITERRATION DIFFERENCE
0.00
0.50
1.00
1.50
2.00
2.50
3.00
3.50
4.00
4.50
ITERRATION NO.

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Question 5:

We are interested in the definition of third-order numerical quadrature in interval (0,1).

a) Determine the minimum number of integration points, and specify the integration points and weights.

Answer

a) The minimum number of integration points (**n**) needed is determined as follows

$$2n - 1 = 3$$

$$n = \frac{3+1}{2} = 2$$

NB. Where n here is the number of the integration points while the number of intervals is 1.

The approximate function used to model the exact integral takes the form of

$$\int_a^b g(x)\,dx \approx C_1 f(x_1) + C_2 f(x_2) = \int_a^b f(x)\,dx$$

Where C1 and C2 are the weights, X1 and X2 are the integration points. While f(x) is the quadrature function used and in this case it is a third order function taking the form of

$$f(x) = \alpha_0 + \alpha_1 x + \alpha_z x^2 + \alpha_3 x^3$$

$$\int_0^1 f(x)\,dx = \alpha_0 x + x_1 \frac{x^2}{2} + \alpha_2 \frac{x^3}{3} + \alpha_3 \frac{x^4}{4}\Bigg|_0^1$$

$$= \alpha_0 + \frac{\alpha_1}{2} + \frac{\alpha_2}{3} + \frac{\alpha_3}{4}$$

$$C_1 f(x_1) + C_2 f(x_2) = C_1(\alpha_0 + \alpha_1 x_1 + \alpha_z {x_1}^2 + \alpha_3 x_1) + C_2(\alpha_0 + \alpha_1 x_2 + \alpha_z {x_2}^2 + \alpha_3 x_2^3)$$

Now we try to manipulate the equation to collect the set of unknowns ($C_1, C_2, x_1 \& x_2$) on the same side

$$\alpha_0(C_1 + C_2) + \alpha_1(C_1x_1 + C_2x_2) + \alpha_2(C_1x_1^2 + C_2x_2^2) + \alpha_3(C_1x_1^3 + C_2x_2^3) = \alpha_0 + \frac{\alpha_1}{2} + \frac{\alpha_2}{3} + \frac{\alpha_3}{4}$$

After that we equate the unknowns $(C_1, C_2, x_1 \& x_2)$ with the values of the coefficients of α_0 α α_2 & α_3

$$C_1 + C_2 = 1$$

$$C_1x_1 + C_2x_2 = \frac{1}{2}$$

$$C_1x_1^2 + C_2x_2^2 = \frac{1}{3}$$

$$C_1x_1^3 + C_2x_2^3 = \frac{1}{4}$$

$$\boldsymbol{X_1} = \mathbf{0.2113} \qquad \boldsymbol{X_2} = \mathbf{0.7886}$$

$$\boldsymbol{C_1} = \frac{\mathbf{1}}{\mathbf{2}} \qquad \boldsymbol{C} = \frac{\mathbf{1}}{\mathbf{2}}$$

b) Is it possible to obtain a third-order quadrature with the following four integration points: x_0 = 1=4, x_1 = 1=2, x_2 = 3=4 and x_3 = 1? If it is possible, compute the corresponding weights; otherwise, justify why not.

Answer

According the (2n+1) rule where n is the number of intervals to which the subdomain is divided, for a 4-integration points n=3 we need at least a polynomial quadrature of 7th order undetermined non equally spaced interval hence because we have 4 integration points X0, X1 X2 and X3 which are unknowns need 4 other weighting coefficients C0, C1, C2 and C3 which are unknowns we can apply **Simpson 3/8** method to this in order to obtain the coefficients corresponding to X0, X1 X2 and X3.

Question 6

a) If n + 1 points Gaussian quadrature is used for numerical integration state the order of the polynomial that is integrated exactly.

Answer

A (2n+3) polynomial is going to be needed.

b) If n = 2 is selected for Gaussian quadrature, which (if any) of the following integrals will be integrated exactly?

Answer

For n=2 (3 integration pts) a 5th order quadrature at least is going to be needed, and in this case if the function that needs to be integrated is a 5th order polynomial or less the answer we are going to obtain is the exact solution.

$$\int_0^1 x^3\,dx \quad \& \quad \int_0^1 x^4\,dx$$

The other 2 functions will not give the exact solution as one of them is not a polynomial (sin x) and the other owns a power which is greater than 5th order so it is can be integrated with an error of

$$E = \Omega f^{(2n+2)''}(n)$$

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Question 7:

Compute $\int_0^1 12x\,dx$, $\int_0^1 5x^3 + 2x\,dx$ **by hand calculation using**

i) Trapezoidal rule over 2 uniform intervals
ii) Simpson's rule over 2 uniform intervals

Compute the error of both approximations. Are the methods behaving as expected?

Answer

For the first integration $\int_0^1 12x\,dx$

Interval points	f(x)
0	0
0.25	3
0.5	6
0.75	9
1	12

	Numerical integration value of trapezoidal rule	Numerical integration value of Simpson rule
Numerical	6	6
TRUE	6	
Error	0	0

For the second integration $\int_0^1 5x^3 + 2x\,dx$

Interval points	f(x)
0	0
0.25	0.578125
0.5	1.625
0.75	3.609375
1	7

	Numerical integration value of trapezoidal rule	Numerical integration value of Simpson rule
Numerical	2.56	2.25
TRUE	2.25	
Error	-0.31	0.00

The 2 methods are behaving exactly as expected. The trapezoidal rule gives the exact results for function 1st order function $\int_0^1 12x\,dx$ as expected as the error of the trapezoidal method is given by the following

$$E = \frac{h^3}{12} f''(x)$$

So it is function in the second derivative of the integrated function $\int_0^1 12x\,dx$ which is Zero, hence the error is going to be zero as well hence obtaing the exact solution. While for the second function $\int_0^1 5x^3 + 2x\,dx$ we have an error corresponding to

$$E = \frac{0.5}{12}(30x)$$

Fort the Simpson method, it gives the exact solution all over the 2 functions as the error of Simpson method is defined as

$$E = \frac{h^5}{90} f^{4'}(x)$$

Which is function in the 4th derivative of the integrated function which is Zero for both functions, hence the error is going to be zero as well hence obtaining the exact solution for both of them.

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Question 10:

Perform the numerical integration of

$$\iint_{0}^{1} (9x^3 + 8x^2)(y^3 + y)\, dx\, dy$$

using Simpson's rule in each direction. Is the approximation behaving as expected?

Answer

Simpson integration is applied for each direction (dx & dy) separately and then the results of both directions is multiplied together to obtain the result of the integration at the end. The exact results are compared to the solution obtained by Simpson with m=16, n=2 intervals

Interval points	f(x)	f(y)
0.000	0.000	0.000
0.063	0.033	0.063
0.125	0.143	0.127
0.188	0.341	0.194
0.250	0.641	0.266
0.313	1.056	0.343
0.375	1.600	0.428
0.438	2.285	0.521
0.500	3.125	0.625
0.563	4.133	0.740
0.625	5.322	0.869
0.688	6.706	1.012
0.750	8.297	1.172
0.813	10.109	1.349
0.875	12.154	1.545
0.938	14.447	1.761
1.000	17.000	2.000

	Num. int. of x	Num. int. of y	Total
	4.917	0.750	3.688
	4.917	0.750	3.688
Error	0.000	0.000	0.000

The method is behaving exactly as expected with zero error as discussed above in question 7.