Solution 7 $\int_0^1 12x\,dx$ and $\int_0^1 (5x^3+2x)\,dx$

(i) Trapezoidal rule over 2 uniform intervals

$a=0, b=1, N=2$

$$I = \int_0^1 12x\,dx$$

For $N=2$: $h = \frac{b-a}{N} = \frac{1-0}{2} = \frac{1}{2}$ so the nodes are 0, 0.5, 1

$x_0 = 0,\ x_1 = 0.5,\ x_2 = 1$

using trapezoidal rule

$$I_1 = \frac{h}{2}\left[f(x_0) + 2\{f(x_1)\} + f(x_2)\right]$$

$$I_1 = \frac{1}{4}\left[0 + 2(12\times 0.5) + 12\right] = 6$$

Exact solution $I = \int_0^1 12x\,dx = 6\ 12\left[\frac{x^2}{2}\right]_0^1 = 6$

Error = |Exact solution − I_1| = |6 − 6| = 0

Thus for a polynomial of degree ≤ 1 (then $f''(x)=0$) and the result implies that error is zero and trapezium rule produces exact results for polynomials of degree ≤ 1.

$$I = \int_0^1 (5x^3+2x)\,dx$$

$a=0,\ b=1,\ N=2$

$h = \frac{b-a}{N} = \frac{1-0}{2} = \frac{1}{2}$

so the nodes are 0, 0.5, 1 i.e. $x_0 = 0,\ x_1 = 0.5,\ x_2 = 1$

using trapezoidal rule

$$I_1 = \frac{h}{2}\left[f(x_0) + 2\{f(x_1) + f(x_2) + \cdots + f(x_{N-1})\} + f(x_N)\right]$$

$$I_1 = \frac{1}{4}\left[f(0) + 2\{f(0.5)\} + f(1)\right] = \frac{1}{4}\left[0 + 2(1.625) + 7\right]$$

$I_1 = 2.5625$

Exact solution: $I = \int_0^1 (5x^3+2x)\,dx = \left[\frac{5x^4}{4} + x^2\right]_0^1 = \frac{5}{4} + 1 = \frac{9}{4}$

$I = 2.25$

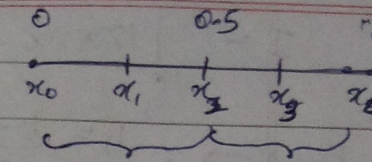
Error = $|I - I_1| = |2.25 - 2.5625| = 0.3125$

(ii) Simpson's rule over 2 uniform intervals

$$I = \int_0^1 12x\,dx$$

~~N=2, a=0, b=1~~

$N=2$ $\cancel{h = \frac{1-0}{2} = \frac{1}{2}}$



Two uniform intervals

$h = \frac{0.5-0}{2} = 0.25$

$$h = \frac{b-a}{N} = \frac{0.5-0}{2} = 0.25$$

So, $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$

$$I = \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) + \frac{h}{3}\left(f(x_2) + 4f(x_3) + f(x_4)\right)$$

$$= \frac{0.25}{3}\left[f(0) + 4f(0.25) + f(0.5)\right] + \frac{0.25}{3}\left[f(0.5) + 4f(0.75) + f(1)\right]$$

$$= \frac{0.25}{3}(0+12+6) + \frac{0.25}{3}(6+36+12)$$

$$= 1.5 + 4.5 = 6$$

Error : $E = -\frac{h^5}{90} f^4(\mu) = 0$

$$I = \int_0^1 (5x^3 + 2x)\,dx$$

$$I = \frac{h}{3}\left[f(0) + 4f(0.25) + f(0.5)\right] + \frac{h}{3}\left[f(0.5) + 4f(0.75) + f(1)\right]$$

$$= \frac{0.25}{3}(0 + 2.3125 + 1.625) + \frac{0.25}{3}(1.625 + 14.4375 + 7)$$

$$= 2.25$$

error = $E = -\frac{h^5}{90} f^4(\mu) = 0$

So the simpson method gave no error ~~for~~ till the third degree polynomial.

Solution 10.
$$I = \int_0^1\int_0^1 (9x^3+8x^2)(y^3+y)\,dx\,dy$$

splitting the integral

$$I_1 = \int_0^1 (9x^3+8x^2)\,dx \quad \text{and} \quad I_2 = \int_0^1 (y^3+y)\,dy$$

Taking intervals $(N) = 2$

So, sub intervals $(n) = 2N = 4$

~~so nodes are 0,~~

Since, $n = 2N = 4$; $h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} = 0.25$

$N = 2$

so nodes are 0, 0.25, 0.5, 0.75, 1.0

$n = 2N = 4$; $I_1 = \frac{h}{3}\left[f(0) + 4\{f(0.25)+f(0.75)\} + 2f(0.5) + f(1)\right]$

$$I_1 = \frac{1}{12}\left[0 + 4\{0.6406 + 8.2968\} + 2(3.125) + 17\right]$$

$$= \frac{1}{12}\left[0 + 35.75 + 6.25 + 17\right] = 4.9167$$

$$I_2 = \int_0^1 (y^3+y)\,dy$$

$n = 2N = 4$; $I_2 = \frac{h}{3}\left[f(0) + 4\{f(0.25)+f(0.75)\} + 2f(0.5) + f(1)\right]$

$$I_2 = \frac{1}{12}\left[0 + 4\{0.2656 + 1.1719\} + 2(0.625) + 2\right]$$

$$= \frac{1}{12}\left[0 + 5.75 + 1.25 + 2\right] = 0.75$$

$$I = I_1 \times I_2 = 4.9167 \times 0.75$$

$$= 3.687525$$

Solution 6 — (a) If $(n+1)$ points Gaussian quadrature is used for numerical integration then the order of polynomial that is exactly integrated is $\cancel{2(n+1)}$.

(b) If $n=2$ is selected for Gaussian quadrature then $2n+1 = 2 \cdot 2+1 = 5$ i.e. uptill 5^{th} degree of polynomials can be exactly integrated so:

(ii) $\int_0^1 x^3 dx$ (iii) $\int_0^1 x^4 dx$ can be exactly integrated.

HW Problem 1 $\quad f(x) = x^3 + 2x^2 + 10x - 20 = 0$

Compute root with 4 iterations.

$x_0 = \sqrt[3]{20}$, $\quad f'(x) = 3x^2 + 4x + 10$

~~Applying Newton Method~~

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{(x_k^3 + 2x_k^2 + 10x_k - 20)}{(3x_k^2 + 4x_k + 10)}$$

$$x_{(k+1)} = \frac{3x_k^3 + 4x_k^2 + \cancel{10x_k} - x_k^3 - 2x_k^2 - \cancel{10x_k} + 20}{(3x_k^2 + 4x_k + 10)}$$

$$x_{(k+1)} = \frac{(2x_k^3 + 2x_k^2 + 20)}{(3x_k^2 + 4x_k + 10)} = \frac{2(x_k^3 + x_k^2 + 10)}{(3x_k^2 + 4x_k + 10)} \cdot \text{ given } x_0 = \sqrt[3]{20}$$

$$x_1 = \frac{2(x_0^3 + x_0^2 + 10)}{(3x_0^2 + 4x_0 + 10)} = 1.7396 \quad \text{and} \quad |x_1 - x_0| = 0.9748$$

$$x_2 = \frac{2(x_1^3 + x_1^2 + 10)}{(3x_1^2 + 4x_1 + 10)} = 1.4049 \quad \text{and} \quad |x_2 - x_1| = 0.3346$$

$$x_3 = \frac{2(x_2^3 + x_2^2 + 10)}{(3x_2^2 + 4x_2 + 10)} = \cancel{0.8133}\ 1.3692 \quad \text{and} \quad |x_3 - x_2| = 0.0357$$

$$x_4 = \frac{2(x_3^3 + x_3^2 + 10)}{(3x_3^2 + 4x_3 + 10)} = 1.3688$$

40
ab = 30
a+b > 4
10×3
6×5
15×2