Polytechnic University of Catalonia

Master on Numerical Methods in Engineering

Homework 1 - FEM

Bruno Aguirre Tessaro

November 10, 2015

Consider the following differential equation

$$-u'' = f \quad \text{in} \quad]0,1[$$

with the boundary conditions $u(0) = 0$ and $u(1) = \alpha$.
The Finite Element discretization is a 2-noded linear mesh given by the nodes $x_i = ih$ for $i = 0, 1, ..., n$ and $h = 1/n$.

1 Find the weak form of the problem. Describe the FE approximation u^h.

The strong form of the problem can be described as:

$$\begin{cases} \frac{d^2u}{dx^2} + Q & = 0 \quad \text{in} \quad]0,1[\\ u(0) = 0 \\ u(1) = \alpha \end{cases}$$

In order to apply the FEM, we need to describe the equation in its weak form. To do so, first, we multiply both sides of the equation by a continuous smooth weight function $w(x)$ and integrate over the domain:

$$\int_0^1 \omega(x) \frac{d^2u(x)}{dx^2} dx + \int_0^1 \omega(x) Q(x) dx = 0$$

Now the first term is integrated by parts in order to decrease its derivative order by 1. Doing this, the weak form of the problem can be represented by:

$$\int_0^1 \frac{d\omega(x)}{dx}\frac{du(x)}{dx}dx = \int_0^1 \omega(x)Q(x)dx + \omega(x)\frac{du(x)}{dx}\Big|_0^1$$

We can approximate the solution of $u(x)$ as a linear combination $u^h(x) = \sum_{i=1}^n N_i(x)u_i$ and by use the Galerkin Method, the weight functions will have the form of $\omega(x)_i = N(x)_i$. Rearranging all the terms and representing the term $\frac{du}{dx}$ by a reaction flux q we have:

$$\int_0^1 \frac{dN_i}{dx}\sum_{j=1}^{n}(\frac{dN_j}{dx}u_j)dx = \int_0^1 N_iQ(x)dx + N_iq\Big|_0^1$$

2 Describe the linear system of equations to be solved.

With the use of Einstein index notation, we can drop the summation and express the equation in the form of:

$$\int_0^1 \frac{dN_i}{dx}\frac{dN_j}{dx}u_jdx = \int_0^1 N_iQ(x)dx + N_iq\Big|_0^1$$

The system of equations will have the form of $K_{ij}u_j = f_i$, with each of this terms represented by:

$$\begin{aligned} K_{ij} &= \int_0^1 \frac{dN_i}{dx}\frac{dN_j}{dx}dx \\ f_i &= \int_0^1 N_iQ(x)dx + N_iq\Big|_0^1 \\ u_j &= u_j \end{aligned}$$

The resulting linear system of equations to be solved can be represented in matrix form as:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & \dots & K_{1n} \\ K_{21} & K_{22} & K_{23} & \dots & K_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ K_{n1} & K_{n2} & K_{n3} & \dots & K_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \dots \\ f_n \end{bmatrix}$$

3 Compute de FE approximation u^h for $n = 3$, $f(x) = sin(x)$ and $\alpha = 3$. Compare it with the exact solution, $u(x) = sin(x) + [3 - sin(1)]x$.

For a 3 finite element discretization, the approximated solution will have the form of:

$$u^h = u_1 N_1 + u_2 N_2 + u_3 N_3 + u_4 N_4$$

The shape functions N_i and their derivatives are represented locally (for each element) by:

$$N_1^e = \frac{x_2 - x}{1/3} \qquad \frac{dN_1^e}{dx} = \frac{-1}{1/3}$$
$$N_2^e = \frac{x - x_1}{1/3} \qquad \frac{dN_2^e}{dx} = \frac{1}{1/3}$$

With the shape functions and its derivatives the stiffness matrix K_{ij} can be assembled, locally, it has the form of:

$$K^e = \frac{1}{1/3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

To evaluate f_i, it is necessary to evaluate the shape functions N_i at each element:

$$N_1^1 = 3(1/3 - x) \qquad N_2^1 = 3(x - 0) \qquad N_1^2 = 3(2/3 - x)$$
$$N_2^2 = 3(x - 1/3) \qquad N_1^3 = 3(1 - x) \qquad N_2^3 = 3(x - 2/3)$$

And f is evaluated as:

$$f_1^1 = \int_0^{1/3} (1 - 3x) sin(x) dx + q_1 = 0.018416 + q_1 \quad f_2^1 = \int_0^{1/3} (3x) sin(x) dx = 0.036627$$
$$f_1^2 = \int_{1/3}^{2/3} (-3x + 2) sin(x) dx = 0.071432 \quad f_2^2 = \int_{1/3}^{2/3} (3x - 1) sin(x) dx = 0.0887638$$
$$f_1^3 = \int_{2/3}^{1} (-3x + 3) sin(x) dx = 0.11658 \quad f_2^3 = \int_{2/3}^{1} (3x - 2) sin(x) dx + q_4 = 0.129 + q_4$$

With this, all the data to mount up the system of equations is evaluated, considering the Dirichlet boundary conditions $u_1 = 0$ and $u_1 = 3$, as:

$$3\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \\ u_3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.018416 + q_1 \\ 0.108059 \\ 0.204218 \\ 0.129 + q_4 \end{bmatrix}$$

The values at the nodes u_1 and u_4 prescribed by the Dirichlet boundary conditions are knowns, leading to a reduction of the system of equations to the form:

$$3\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.108059 \\ 0.204218 + (3 * 3) \end{bmatrix}$$

Solving the system we find $u_2 = 1.046704$ and $u_3 = 2.057388$. This will lead to a approximation of the form:

$$u^h = 1.046704N_2 + 2.057388N_3 + 3N_4$$

And the reaction fluxes can now be calculated as follows:

$$3(-1.046704) = 0.018416 + q_1 \qquad q_1 = -3.158527$$
$$3(-2.057388 + 3) = 0.129 + q_4 \qquad q_4 = 2.698835$$

Now we can plot the analytical solution $u(x) = sin(x) + [3 - sin(1)]x$ and the approximated solution $u^h = 1.046704N_2 + 2.057388N_3 + 3N_4$ to make a comparison of the results.

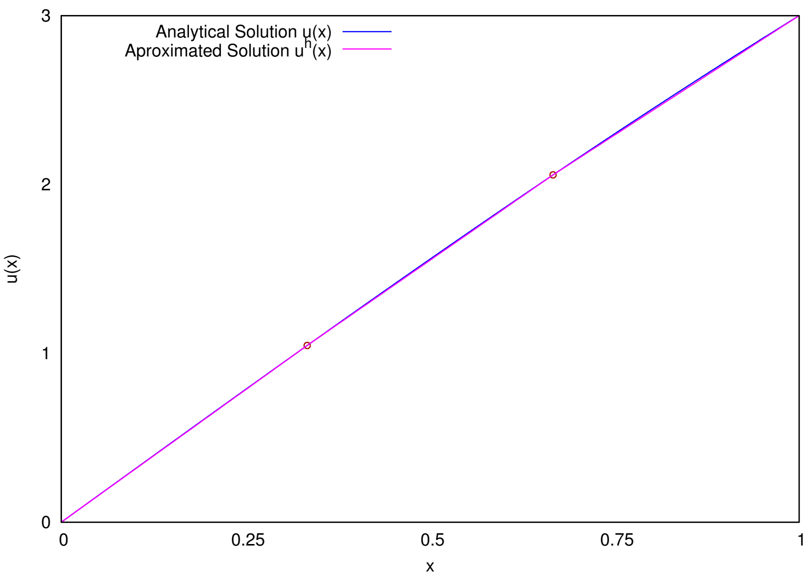

Figure 3.1: Comparison between analytical and FEM approximated solution

We can observe that the solutions have great agreement with each other showing that the FE approximation is satisfactory.