Finite Elements

Plane Elasticity (Homework 2)

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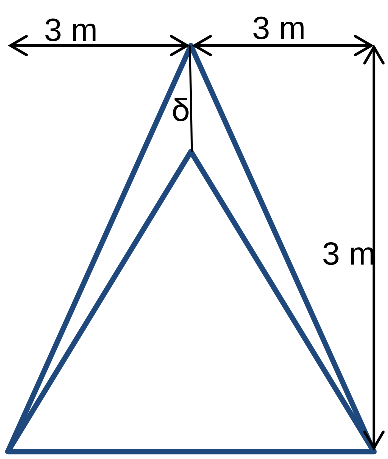
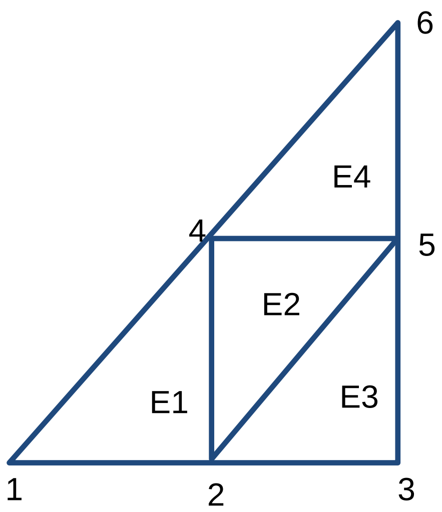
Given, a triangular thin plate is deformed under its self-weight and an imposed vertical displacement δ on the tip. A plane stress model is used to analyze the structural response of the plate. The thickness is assumed to be equal to 1, i.e. t = 1 m. Using the symmetry of the problem, only the left half of the domain is analyzed.

The finite element discretization with a mesh of 4 3-noded linear triangular elements and 6 nodes is shown in figure below.

Given $E = 10\text{GPa}, v = 0.2,\ \rho g = 10^3$ N/m^2 and δ=0.01m





1. The strong form of the given problem can be analyzed following the assumption of 2D plane elasticity problem which is:

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = 0 \qquad \boldsymbol{x} \in \boldsymbol{\Omega}$$

Where,

$$\boldsymbol{b} = \rho \cdot \boldsymbol{g}$$

The appropriate boundary conditions are:

$$x:\ u1 = u2 = u3 = u5 = u6 = 0\ m$$

$$y: \begin{cases} v6 = 10^{-2}\ m \\ v1 = v2 = v3 = 0\ m \end{cases}$$

2. Table 1 and table 2 show the nodal coordinates X and the connectivity matrix T respectively

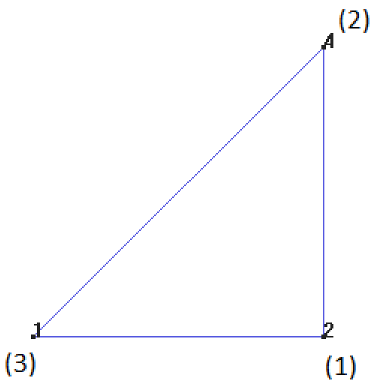
Table 1:Nodal coordinates.

Node	X	Y
1	-3	0
2	-1.5	0
3	0	0
4	-1.5	1.5
5	0	1.5
6	0	3

Table 2: Connectivity matrix.

Element	Nodes		
1	2	4	1
2	4	2	5
3	3	5	2
4	5	6	4

Description of mesh: From the above figure we can see that we have four elements. In order to make the descritization easier, local numbering is made such that in every element the node in the right angle vertex has a local number equal to 1 as shown in fig below.



1,2,4- Global numbering
1,2,3- Local numbering

. The approximate solution of the displacement in x and y directions can be expressed as:

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$
$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

Where N_i are the shape function and (u_i,v_i) are node displacement in x and y directions.

The equation can be written as:

$$\boldsymbol{u} = \boldsymbol{N} \cdot \boldsymbol{a}^e$$

N and $\mathbf{a}^{(e)}$ contain as many matrices N_i and vectors $a^{(e)}$ as element nodes.

The expression of the shape function is found to be as:

$$N_i = \frac{1}{2 \cdot A^e}(a_i + b_i x + c_i y)$$

Where A^e is the area of the element and

$$a_i = x_j y_k - x_k y_j$$
$$b = y_l - y_k$$
$$c = x_k - x_j$$

The shape function takes the value 1 at node i and zero at the other two nodes.

Strain and stress are obtained as

$$\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{a}^e$$

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{B}\boldsymbol{a}^e$$

Where **B** contains as many matrices as element nodes and is obtained as:

$$B_i = \frac{1}{2A^e}\begin{bmatrix} b_i & 0 \\ 0 & c_i \\ c_i & b_i \end{bmatrix}$$

And **D** is the constitutive matrix.

In this case, as the problem is a *plane stress* and the material is *isotropic*, the constitutive matrix is:

$$\boldsymbol{D} = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{12} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

For isotropic elasticity plan stress problem we have,

$$d_{11} = d_{22} = \frac{E}{1-v^2} = \frac{10}{1-.2^2} = 10.417 \text{ Gpa}$$

$$d_{12} = d_{21} = v d_{11} = 2.083\ GPa$$

$$d_{33} = \frac{E}{2(1+v)} = \frac{10}{2(1+0.2)} = 4.167 GPa$$

The discretized equilibrium equations for the 3-noded triangle will be derived by applying the PVW. In FEM the equilibrium of the forces is apply at the nodes only, so a nodal point load will be defined in order to balance the external forces and the internal forces due to the element deformation.

For each individual element, the equilibrating nodal forces are obtained as:

$$\iint_{A^e} \delta \boldsymbol{\varepsilon}^{\boldsymbol{T}} \boldsymbol{\sigma} t dA = \iint_{A^e} \delta \boldsymbol{u}^{\boldsymbol{T}} \boldsymbol{b} t dA + \oint_{l^e} \delta \boldsymbol{u}^{\boldsymbol{T}} \boldsymbol{t} t ds + \sum_{i=1}^{3} \delta u_i U_i + \sum_{i=1}^{3} \delta v_i V_i$$

Where U and V are the equilibrating nodal forces.

After the interpolation of the virtual displacement in terms of the nodal values, substituting in the previous equation and taking into account that the virtual displacement is arbitrary, we obtain

$$[\iint_{A^e} \boldsymbol{B}^{\boldsymbol{T}} \boldsymbol{D} \boldsymbol{B} t dA] \boldsymbol{a}^{\boldsymbol{e}} - \iint_{A^e} \boldsymbol{B}^{\boldsymbol{T}} \boldsymbol{E} \boldsymbol{\varepsilon}^{\boldsymbol{0}} t dA + \iint_{A^e} \boldsymbol{B}^{\boldsymbol{T}} \boldsymbol{\sigma}^{\boldsymbol{0}} t dA$$

$$- \iint_{A^e} \boldsymbol{N}^{\boldsymbol{T}} \boldsymbol{b} t dA \ - \oint_{l^e} \boldsymbol{N}^{\boldsymbol{T}} \boldsymbol{t} t dS = \boldsymbol{q}^{\boldsymbol{e}}$$

Where $\mathbf{q}^{\mathbf{e}}$ is the equilibrating nodal forces in terms of the nodal forces due to the element deformation (first three integrals) the body forces (second integral) and the surface traction(third integral) and $\mathbf{a}^{\mathbf{e}}$ is the nodal displacement.

The global equilibrium equation could be written as

$$\boldsymbol{Ka} = \boldsymbol{f} \text{ ....}\textbf{(i)}$$

Where **K** is the element stiffness matrix and it can be written for a 3 nodes triangular element as

$$\boldsymbol{K}_{ij}^{e} = \left(\frac{t}{4A}\right)^{e} \begin{bmatrix} b_i b_j d_{11} - c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ c_i b_j d_{21} + b_i c_j d_{33} & b_i b_j d_{33} + c_i c_j d_{22} \end{bmatrix}$$

a is the displacement matrix, given by

$$a_i^e = \begin{pmatrix} u_i \\ v_i \end{pmatrix}$$

$$a = (a1, a2 \ldots . an)^T$$

And the body forces equally distributed for a 3 nodes triangular element can be computed as

$$f_{bi}^{e} = \frac{(At)^{e}}{3} \begin{pmatrix} b_x \\ b_y \end{pmatrix}$$

In order to set up the linear system of equations, first of all, we compute the constitutive matrix **D**

$$\boldsymbol{D} = 10^9 \begin{bmatrix} 10.417 & 2.083 & 0 \\ 2.083 & 10.417 & 0 \\ 0 & 0 & 4.167 \end{bmatrix}$$

and the element strain matrix **B** for the elements 1, 3 and 4

$$\boldsymbol{B}^{(\mathbf{1\ 3\ 4})} = \begin{bmatrix} -1.5 & 0 & 1.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.5 & 0 & 1.5 \\ 0 & -1.5 & -1.5 & 1.5 & 1.5 & 0 \end{bmatrix}$$

And for the element 2

$$\boldsymbol{B}^{(2)} = \begin{bmatrix} 0 & 0 & 1.5 & 0 & -1.5 & 0 \\ 0 & -1.5 & 0 & 0 & 0 & 1.5 \\ -1.5 & 0 & 0 & 1.5 & 1.5 & -1.5 \end{bmatrix}$$

Then we compute the **K** matrix for each element. K matrix is same for elements 1,3 and 4.Stiffness matrix for element 2 is different which is computed below.

$\boldsymbol{K}^{1\,3\,4}$ **:**

5,21E+09	0	-5,2E+09	1,04E+09	0	-1E+09
0	2,08E+09	2,08E+09	-2,1E+09	-2,1E+09	0
-5,2E+09	2,08E+09	7,29E+09	-3,1E+09	-2,1E+09	1,04E+09
1,04E+09	-2,1E+09	-3,1E+09	7,29E+09	2,08E+09	-5,2E+09
0	-2,1E+09	-2,1E+09	2,08E+09	2,08E+09	0
-1E+09	0	1,04E+09	-5,2E+09	0	5,21E+09

$\boldsymbol{K}^2$ **:**

7,29E+09	-3,1E+09	-2,1E+09	1,04E+09	-5,2E+09	2,08E+09
-3,1E+09	7,29E+09	2,08E+09	-5,2E+09	1,04E+09	-2,1E+09
-2,1E+09	2,08E+09	2,08E+09	0	0	-2,1E+09
1,04E+09	-5,2E+09	0	5,21E+09	-1E+09	0
-5,2E+09	1,04E+09	0	-1E+09	5,21E+09	0
2,08E+09	-2,1E+09	-2,1E+09	0	0	2,08E+09

Due to a symmetricity property of the matrix we have:

$$K_{12} = K_{21}; K_{13} = K_{31}; K_{14} = K_{41};\ K_{15} = K_{51}; K_{16} = K_{61}$$

$$K_{23} = K_{32}; K_{24} = K_{42}; K_{25} = K_{52}; K_{26} = K_{62}$$

$$K_{34} = K_{43}; K_{35} = K_{53}; K_{36} = K_{63}$$

$$K_{45} = K_{54}; K_{46} = K_{64}; K_{56} = K_{65};$$

The global stiffness matrix is given by:

$$K_{global} = \sum_{i=1}^{4} K_{i_{global}}$$

Where K_{global} is

5,21E+09	0	-5,2E+09	1,04E+09	0	0	0	-1E+09	0	0	0	0
0	2,08E+09	2,08E+09	-2,1E+09	0	0	-2,1E+09	0	0	0	0	0
-5,2E+09	2,08E+09	1,46E+10	-3,1E+09	-5,2E+09	1,04E+09	-4,2E+09	3,13E+09	0	-3,1E+09	0	0
1,04E+09	-2,1E+09	-3,1E+09	1,46E+10	2,08E+09	-2,1E+09	3,13E+09	-1E+10	-3,1E+09	0	0	0
0	0	-5,2E+09	2,08E+09	7,29E+09	-3,1E+09	0	0	-2,1E+09	1,04E+09	0	0
0	0	1,04E+09	-2,1E+09	-3,1E+09	7,29E+09	0	0	2,08E+09	-5,2E+09	0	0
0	-2,1E+09	-4,2E+09	3,13E+09	0	0	1,46E+10	-3,1E+09	-1E+10	3,13E+09	0	-1E+09
-1E+09	0	3,13E+09	-1E+10	0	0	-3,1E+09	1,46E+10	3,13E+09	-4,2E+09	-2,1E+09	0
0	0	0	-3,1E+09	-2,1E+09	2,08E+09	-1E+10	3,13E+09	1,46E+10	-3,1E+09	-2,1E+09	1,04E+09
0	0	-3,1E+09	0	1,04E+09	-5,2E+09	3,13E+09	-4,2E+09	-3,1E+09	1,46E+10	2,08E+09	-5,2E+09
0	0	0	0	0	0	0	-2,1E+09	-2,1E+09	2,08E+09	2,08E+09	0
0	0	0	0	0	0	-1E+09	0	1,04E+09	-5,2E+09	0	5,21E+09

For the given problem, displacement matrix a is:

$$a = (a1, a2, a3, a4, a5, a6)^T$$

And the global vector body forces will be computed as:

$$f_{global} = (0, f_b, 0, 3f_b, 0, f_b, 0, 3f_b, 0, 3f_b, 0, f_b)^T$$

The body forces vector, due to the body is deformed under its self-weight, has only a vertical component b_y.

$$f_{bi}^{e} = \frac{(9/8)}{3}\begin{pmatrix} 0 \\ -1000 \end{pmatrix}$$

The global vector body force is:

$$f_{global} = (0, -375, 0, -1125, 0, -375, 0, -1125, 0, -1125, 0, -375)^T$$

Putting in equation 1 and solving the matrix we get,

$$1.4583e10 \cdot u_4 - 3.125e9 \cdot v_4 + 3.125e9 \cdot v_5 = -10417e7$$

$$-3.125e9 \cdot u_4 + 1.4583e10 \cdot v_4 - 4.1667e9 \cdot v_5 = -1125$$

$$3.125e9 \cdot u_4 - 4.1667e9 \cdot v_4 + 1.4583e10 \cdot v_5 = -5.208e$$

<u>Results</u>

The FEM approximation gives us the following results.

Table 1: Displacement [m]

Node	Displacement (x)	Displacement (y)
1	0	0
2	0	0
3	0	0
4	-1.282e-4	-1.133e-3
5	0	-3.867e-3
6	0	0.01

Hence, $u_4 = -1.29 \times 10^{-4}$; $v_4 = -1.13 \times 10^{-3}$; $v_5 = -3.87 \times 10^{-3}$