Finite Elements Homework 2

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1 Strong form and Boundary Conditions

We can express the following governing equations:

Kinematic equations

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{xz} = 0$$

Knowing that it is a plane stress study, we can extract the following constitutive matrix:

$$D = \begin{bmatrix} \frac{E}{1-\nu} & \nu\frac{E}{1-\nu} & 0 \\ \nu\frac{E}{1-\nu} & \frac{E}{1-\nu} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix}$$

Which has to fulfill the following :

Constitutive equations

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \frac{E}{1-\nu} & \nu\frac{E}{1-\nu} & 0 \\ \nu\frac{E}{1-\nu} & \frac{E}{1-\nu} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

And finally also requires to fullfill the balance- equilibrium equations, which can be expressed as follows:

Balance equations

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2 Mesh, X, T, Local Numbering

We define the requested matrices as:

$$X = \begin{bmatrix} 0 & 0 \\ 1.5 & 0 \\ 3 & 0 \\ 1.5 & 1.5 \\ 3 & 1.5 \\ 3 & 3 \end{bmatrix} \qquad T = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

And the extra connectivity array with the local number for each of the elements:

$$CA = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 2 & 5 \\ 3 & 5 & 2 \\ 5 & 6 & 4 \end{bmatrix}$$

3 Linear system of equations

We know that each K^e matrix that we compute will be a 6x6 matrix, due to the 2 DOF of each node. The global matrix will be a 12x12 one. So, after some work, our global stiffness matrix and force vector will have the following structure:

$$K = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & K_{13}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{12}^{(1)} + K_{11}^{(2)} + K_{11}^{(3)} & K_{12}^{(3)} & K_{23}^{(1)} + K_{11}^{(2)} & K_{13}^{(2)} + K_{13}^{(3)} & 0 \\ 0 & K_{21}^{(3)} & K_{22}^{(3)} & 0 & K_{23}^{(3)} & 0 \\ K_{31}^{(1)} & K_{32}^{(1)} + K_{22}^{(2)} & 0 & K_{33}^{(1)} + K_{21}^{(2)} + K_{11}^{(4)} & K_{23}^{(2)} + K_{12}^{(4)} & K_{13}^{(4)} \\ 0 & K_{32}^{(2)} + K_{31}^{(3)} & K_{32}^{(3)} & K_{31}^{(2)} + K_{21}^{(4)} & K_{33}^{(2)} + K_{33}^{(3)} + K_{22}^{(4)} & K_{23}^{(4)} \\ 0 & 0 & 0 & K_{31}^{(4)} & K_{32}^{(4)} & K_{33}^{(4)} \end{bmatrix}$$

$$f = \begin{bmatrix} f_1^{(1)} + r_1 \\ f_2^{(1)} + f_1^{(2)} + f_1^{(3)} + r_2 \\ f_2^{(3)} + r_3 \\ f_3^{(1)} + f_2^{(2)} + f_1^{(4)} \\ f_3^{(2)} + f_3^{(3)} + f_2^{(4)} \\ f_3^{(4)} \end{bmatrix}$$

And we must solve the following system:

$$Ka = f$$

4 FE aproxximation

To set up the linear system of equations for the discretization in figure 2 we need to compute first the stiffness matrix and the force vector for each element. Then, we must assemble the global stiffness matrix.

The stiffness matrix for an element e is computed this way:

$$K^e = \int\int_{A^{(e)}} \begin{bmatrix} B_1^T \\ B_2^T \\ B_3^T \end{bmatrix} D \begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix} tdA$$

We already know t=1 and D. We must compute B using the following procedure:

$$B_i = \frac{1}{2A} \begin{bmatrix} b_i & 0 \\ 0 & c_i \\ c_i & b_i \end{bmatrix} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix}$$

Where the coefficients can be computed as:

$$a_i = x_j y_k - x_k y_j$$

$$b_i = y_j - y_k$$

$$c_i = x_k - x_j$$

As we are working on local coordinates, we will have two different K matrices. One for the elements 1, 3, 4 and another one for the element number 2.

Elements 1, 3, 4

$$a_1 = 0 \qquad a_2 = 0 \qquad a_3 = 1.125$$

$$b_1 = 1.5 \qquad b_2 = 0 \qquad b_3 = -1.5$$

$$c_1 = -1.5 \qquad c_2 = 1.5 \qquad c_3 = 0$$

$$B = \frac{2}{3}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & -1 \end{bmatrix}$$

Element 2

$$a_1 = 1.125 \qquad a_2 = 0 \qquad a_3 = 0$$

$$b_1 = -1.5 \qquad b_2 = 0 \qquad b_3 = 1.5$$

$$c_1 = 1.5 \qquad c_2 = -1.5 \qquad c_3 = 0$$

$$B = \frac{2}{3}\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

The same fill happen for the force vectors. In our case, we will only have force vector due to gravity (body forces), in the y- direction. It is computed as follows:

$$f_b^e = \int\int_{A^{(e)}} N^T btdA = \frac{(At)^e}{3}\begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

If we start computing:

$$K^{(1)} = K^{(2)} = K^{(3)} = K^{(4)} = 10^9 \begin{bmatrix} 9.375 & -4.375 & -3.125 & 1.25 & -6.25 & 3.125 \\ & 9.375 & 3.125 & -6.25 & 1.25 & -3.125 \\ & & 3.125 & 0 & 0 & -3.125 \\ & & & 6.25 & -1.25 & 0 \\ & symm & & & 6.25 & 0 \\ & & & & & 3.125 \end{bmatrix}$$

$$f^{(1)} = f^{(2)} = f^{(3)} = f^{(4)} = \begin{bmatrix} 0 \\ -375 \end{bmatrix}$$

We have used MATLAB to compute the solution of the full u^h approximation.

The global matrix, complete: (1e10x scaling)

	1	2	3	4	5	6	7	8	9	10	11	12
1	9.3750	-4.3750	-3.1250	1.2500	0	0	-6.2500	3.1250	0	0	0	0
2	-4.3750	9.3750	3.1250	-6.2500	0	0	1.2500	-3.1250	0	0	0	0
3	-3.1250	3.1250	18.7500	-4.3750	-3.1250	1.2500	-6.2500	-1.8750	-6.2500	1.8750	0	0
4	1.2500	-6.2500	-4.3750	18.7500	3.1250	-6.2500	1.8750	-3.1250	-1.8750	-3.1250	0	0
5	0	0	-3.1250	3.1250	3.1250	0	0	0	0	-3.1250	0	0
6	0	0	1.2500	-6.2500	0	6.2500	0	0	-1.2500	0	0	0
7	-6.2500	1.2500	-6.2500	1.8750	0	0	25	-8.7500	-6.2500	2.5000	-6.2500	3.1250
8	3.1250	-3.1250	-1.8750	-3.1250	0	0	-8.7500	21.8750	6.2500	-12.5000	1.2500	-3.1250
9	0	0	-6.2500	-1.8750	0	-1.2500	-6.2500	6.2500	12.5000	0	0	-3.1250
10	0	0	1.8750	-3.1250	-3.1250	0	2.5000	-12.5000	0	15.6250	-1.2500	0
11	0	0	0	0	0	0	-6.2500	1.2500	0	-1.2500	6.2500	0
12	0	0	0	0	0	0	3.1250	-3.1250	-3.1250	0	0	3.1250

Reduced:

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0	0	0	0	0
4	0	0	0	1	0	0	0	0	0	0	0	0
5	0	0	0	0	1	0	0	0	0	0	0	0
6	0	0	0	0	0	1	0	0	0	0	0	0
7	0	0	0	0	0	0	2.5000e+10	-8.7500e+09	0	2.5000e+09	0	0
8	0	0	0	0	0	0	-8.7500e+09	2.1875e+10	0	-1.2500e+10	0	0
9	0	0	0	0	0	0	0	0	1	0	0	0
10	0	0	0	0	0	0	2.5000e+09	-1.2500e+10	0	1.5625e+10	0	0
11	0	0	0	0	0	0	0	0	0	0	1	0
12	0	0	0	0	0	0	0	0	0	0	0	1

Global force vector:

	1
1	0
2	-375
3	0
4	-1125
5	0
6	-375
7	0
8	-1125
9	0
10	-1125
11	0
12	-375

Reduced force vector: (after reorder due to enforced displacements)

	1
1	0
2	0
3	0
4	0
5	0
6	0
7	312500000
8	-312501125
9	0
10	-1125
11	0
12	-0.0100

Then we only need to solve the system, obtaining the following solution for the displacements:

	1
1	0
2	0
3	0
4	0
5	0
6	0
7	0.0065
8	-0.0226
9	0
10	-0.0191
11	0
12	-0.0100

This solution makes sense because displacements are also affected by body forces, giving a solution with some displacements greater than the enforced one.