Finite Element

Home work 1

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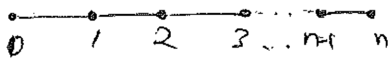
Computational Mechanics

Gov. eq: $-u'' = f$, $x \in [0, 1]$

B.C: $u(0) = \cancel{u}\ 0$, $u(1) = X$.



1.) Weak form:

$$-\frac{d^2u}{dx^2} = f$$

$$\Rightarrow \int_0^1 w\left(\frac{d^2u}{dx^2} + f\right) dx = 0$$

where w is the weighting function,

$$\Rightarrow -\int_0^1 w \frac{d^2u}{dx^2}\, dx = \int_0^1 wf\, dx$$

Integrating L.H.S. by parts, we have:

$$\Rightarrow \cancel{\int} -\left[w \frac{du}{dx}\right]_0^1 + \int \frac{dw}{dx} \cdot \frac{du}{dx}\, dx = \int_0^1 wf\, dx$$

$$\Rightarrow \int \frac{dw}{dx} \cdot \frac{du}{dx}\, dx = w\frac{du}{dx}\Big|_{x=1} - w\frac{du}{dx}\Big|_{x=0} + \cancel{\int_0^1 f\, dx} + \int_0^1 w f\, dx$$

$\frac{du}{dx}$ at boundary are the unknown fluxes,

so, $q_0 = -\frac{du}{dx}\Big|_{x=0}$, $q_n = -\frac{du}{dx}\Big|_{x=1}$

$$\Rightarrow \int \frac{dw}{dx} \cdot \frac{du}{dx}\, dx = -Wq_n + Wq_0 + \int f w\, dx \leftarrow \boxed{\text{Weak form}}$$

2) Approximating u by ~~a~~ shape functions, we get

$$u \simeq u^h = \sum_{i=0}^{n} N_i u_i$$

where N_i is the shape function defined ~~as~~ such that $N_i = 1$ at $x = x_i$ and linearly decreasing to ~~a~~ 0 at adjacent nodes.

So, $N_i = 0$ at $x = x_{i+1}$, $x = x_{i-1}$.

~~for node 0~~

We use $W_j = N_j$ are weighting functions, ~~for~~ for $j = 0$ to n.

$\underline{j=0}$ $\quad W_0 = N_0$

$$\cancel{\int \frac{dW}{dx}} \int \frac{dN_0}{dx} \cdot \sum_{i=0}^{n} u_i \frac{dN_i}{dx} dx = \left[N_0 q_0\right]_{x=1} - \left[N_0 q\right]_{x=0} + \int_0^1 N_0 f \, dx$$

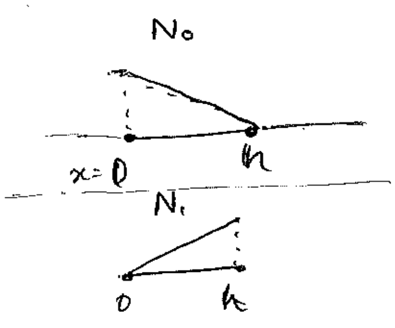
$N_0(x=1) = 0$, so, $\left[N_0 q\right]_{x=1} = 0$.

~~also, $N_0 = 0$ at $x = h$ to 1, so,~~

$$\Rightarrow \int_0^1 \frac{dN_0}{dx} \sum_{i=0}^{n} u_i \frac{dN_i}{dx} dx = -N_0 q_0 + \int_0^h N_0 f \, dx$$

$$\Rightarrow \sum_{i=0}^{n} \left(\int_0^1 \frac{dN_0}{dx} \frac{dN_i}{dx} dx \right) u_i = -N_0 q_0 + \int_0^1 N_0 f \, dx$$

since, $N_0 = \begin{cases} 1 - \dfrac{x}{h} & 0 < x < h \\ 0 & x \geq h \end{cases}$



~~and~~ and $\dfrac{dN_0}{dx} = \begin{cases} \dfrac{-1}{h} & 0 < x < h \\ 0 & x \geq h \end{cases}$

~~sol~~ we can write the integrals as,

$$\Rightarrow \sum_{i=0}^{n} \left[\int_0^h \left(\frac{dN_0}{dx}\right) \frac{dN_i}{dx}\, dx \right] u_i = -q_0 + \int_0^h N_0 f\, dx$$

from $x=0$ to h, only N_0 & N_1 are non-zero. So,

$$\Rightarrow \int_0^h \frac{dN_0}{dx}\left(\frac{dN_0}{dx} u_0 + \frac{dN_1}{dx} u_1 \right) dx = -q_0 + \int_0^h N_0 f\, dx$$

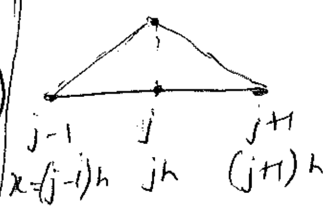
$$\Rightarrow \underbrace{\left[\int_0^h \frac{dN_0}{dx} \cdot \frac{dN_0}{dx}\, dx \right]}_{K_{00}} u_0 + \underbrace{\left[\int_0^h \frac{dN_0}{dx} \cdot \frac{dN_1}{dx}\, dx \right]}_{K_{01}} u_1 = \underbrace{-q_0 + \int_0^h N_0 f\, dx}_{f_0}$$

$$K_{00} = \int_0^h \left(\frac{-1}{h}\right)^2 dx = \frac{1}{h}$$

$$K_{01} = \int_0^h \frac{-1}{h} \cdot \frac{1}{h}\, dx = -\frac{1}{h}$$

for node j

$$N_j = \begin{cases} 1 + \dfrac{x - jh}{h} & x \in ((j-1)h, \cancel{(j+1)h}\ jh) \\ 1 - \dfrac{x - jh}{h} & x \in (jh, (j+1)h) \\ 0 & \text{otherwise.} \end{cases}$$



node j

$$\& \quad \frac{dN_j}{dx} = \begin{cases} \dfrac{1}{h} & (j-1)h \leq x < jh \\ \dfrac{-1}{h} & jh < x < (j+1)h \\ 0 & \text{otherwise} \end{cases}$$

hence, the integral can be reduced to,

$$\int_{(j-1)h}^{(j+1)h} \frac{dN_j}{dx}\left(\frac{dN_{j-1}}{dx} u_{j-1} + \frac{dN_j}{dx} u_j + \frac{dN_{j+1}}{dx} u_{j+1}\right) dx$$

$$= \int_{(j-1)h}^{(j+1)h} N_j f \, dx$$

$$\Rightarrow \underbrace{\left[\int_{x_{j-1}}^{x_{j+1}} \frac{dN_j}{dx}\frac{dN_{j-1}}{dx}\, dx\right]}_{K_{j,j-1}} u_{j-1} + \underbrace{\left[\int_{x_{j-1}}^{x_{j+1}} \left(\frac{dN_j}{dx}\right)^2 dx\right]}_{K_{jj}} u_j + \underbrace{\left[\int_{x_{j-1}}^{x_{j+1}} \frac{dN_j}{dx}\frac{dN_{j+1}}{dx}\, dx\right]}_{K_{j,j+1}} u_{j+1}$$

$$= \int_{x_{j-1}}^{x_{j+1}} N_j f \, dx$$

$$K_{jj-1} = K_{j,j+1} = \frac{-1}{h}, \quad K_{jj} = \frac{2}{h}$$

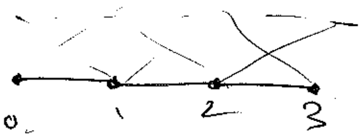
$$f_j = \int_{x_{j-1}}^{x_{j+1}} N_j f_j \, dx$$

$||^{rly}$, for node n, the equation will be,

$$\Rightarrow K_{n,n-1} = \frac{-1}{h}, \quad K_{nn} = \frac{1}{h}, \quad f_n = q_n + \int_{x_{n-1}}^{x_n} N_n f \, dx$$

3.) <u>for n=3</u>

$$\bar{K} = \frac{1}{h}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$



$$\bar{u} = \begin{bmatrix} u_0 & u_1 & u_2 & u_3 \end{bmatrix}^T$$

$$\bar{f} = \begin{bmatrix} -q_0 + \int_0^h N_0 f\,dx \\ \int_0^{2h} N_1 f\,dx \\ \int_h^{3h} N_2 f\,dx \\ q_4 + \int_{2h}^{3h} N_3 f\,dx \end{bmatrix} = \begin{bmatrix} -q_0 + 0.018616 \\ 0.108058 \\ 0.204222 \\ q_3 + 0.129001 \end{bmatrix}$$

using $h = 1/3$, the above integrals are evaluated.

After shuffling the knowns to RHS & unknowns to LHS we get, the following equations:

$\cancel{-q_0 = 1 \cdot u_0}$

$$+q_0 - \frac{u_1}{h} = 0.018616$$

$$2\frac{u_1}{h} - \frac{u_2}{h} = 0.108058$$

$$-\frac{u_1}{h} + \frac{2u_2}{h} = \frac{\alpha}{h} + 0.204222 = 9.204222$$

$$-\frac{u_2}{h} - q_3 = -\frac{\alpha}{h} + 0.129001 = -0.870999 \approx -0.871$$

On solving the above eq^ns, we get

$u_1 = 1.046706$, $u_2 = 2.057388$

$q_0 = 3.15852$, $q_3 = -5.30116$

On comparison with the analytical solution, the error found is:

Node 1: 3.52083×10^{-7}

Node 2: 2.3324×10^{-7}