Finite Elements Homework 1

Daniel Díez Rodríguez

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Consider the following differential equation:

$$u'' = f \tag{1}$$

with the boundary conditions $u(0) = 0$ y $u(1) = \alpha$.

The Finite Element discretization is a 2-noded linear mesh given by $x_i = ih$ for $i = 0, 1, ..., n$ and $h = 1/n$.

1. Find the weak form of the problem. Describe the FE approximation u^h.
2. Describe the linear system of equations to be solved.
3. Compute the FE approximation u^h for $n = 3$, $f(x) = sin(x)$ and $\alpha = 3$. Compare it with the exact solution $u(x) = sin(x) + (3 - sin(1))x$.

Our function is:

$$\frac{\partial}{\partial x}\left(\frac{\partial u(x)}{\partial x}\right) + f(x) = 0 \tag{2}$$

And the boundary conditions are:

$$\phi - \overline{\phi} = 0 \tag{3}$$

at x=0 and x=1

We multiply by the test function $v(x)$ and integrate.

$$\int_0^1 v(x)\left(\frac{\partial}{\partial x}\left(\frac{\partial u(x)}{\partial x}\right) + f(x)\right)\,dx = \int_0^1 v(x)f(x) \tag{4}$$

$$\int_0^1 v(x)\left(\frac{\partial}{\partial x}\left(\frac{\partial u(x)}{\partial x}\right) + f(x)\right)\,dx = \int_0^1 v(x)(\frac{\partial}{\partial x}\left(\frac{\partial u(x)}{\partial x}\right)\,dx + \int_0^1 v(x)f(x))\,dx \tag{5}$$

First integral can be computed by parts.

$$v(x)\frac{\partial u(x)}{\partial x}\Big]_0^1 - \int_0^1 \frac{\partial u(x)}{\partial x}\frac{\partial v(x)}{\partial x}\,dx = \int_0^1 v(x)f(x)\,dx \tag{6}$$

Using Galerkin method:

$$v_i(x) = N_i(x) \tag{7}$$

$$u(x) \simeq u^h(x) = \sum_{i=0}^{n} N_i(x)a_i \tag{8}$$

Applying these relationships to (5) we obtain:

$$\int_0^1 \frac{\partial N_i(x)}{\partial x}\frac{\partial N_j(x)a_j}{\partial x}\,dx = \int_0^1 N_i(x)f(x)\,dx + N_i q\Big]_0^1 \tag{9}$$

We can write this equation as:

$$K_{ij}a_j = f_i \tag{10}$$

$$\begin{pmatrix} K_{00} & K_{01} & K_{02} & K_{03} \\ K_{10} & K_{11} & K_{12} & K_{13} \\ K_{20} & K_{21} & K_{22} & K_{23} \\ K_{30} & K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

where

$$K_{ij} = \int\limits_0^1 \frac{\partial N_i(x)}{\partial x}\frac{\partial N_j(x)}{\partial x}\,dx \tag{11}$$

$$f_i = \int\limits_0^1 N_i(x)f(x)\,dx + N_i q\Big]_0^1 \tag{12}$$

Now we have to compute the FE approximation u^h for $n = 3$, $f(x) = sin(x)$ and $\alpha = 3$.

The bar is divided in three elements of the same length: $l^e = 1/3$. We can obtain the stifness matrix and the f-vector for every element and then build the global matrix.

$$K_{ij}^e = \int\limits_{l^e} \frac{\partial N_i^e(x)}{\partial x}\frac{\partial N_j^e(x)}{\partial x}\,dx \tag{13}$$

$$f_i^e = \int\limits_{l^e} N_i^e(x)f(x)\,dx \tag{14}$$

$$K_{ij} = \begin{pmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 \\ K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & 0 \\ 0 & K_{21}^2 & K_{22}^2 + K_{11}^3 & K_{12}^3 \\ 0 & 0 & K_{21}^3 & K_{22}^3 \end{pmatrix}$$

$$f_i = \begin{pmatrix} f_1^1 + q_1 \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ f_2^3 + q_4 \end{pmatrix}$$

We also have to define the N_i functions and their derivatives for each element.

$N_1^1 = \frac{x_2 - x}{l}$; $\frac{dN_1^1}{dx} = -\frac{1}{l}$

$N_2^1 = \frac{x - x_1}{l}$; $\frac{dN_1^1}{dx} = \frac{1}{l}$

$N_1^2 = \frac{x_3 - x}{l}$; $\frac{dN_1^1}{dx} = -\frac{1}{l}$

$N_2^2 = \frac{x - x_2}{l}$; $\frac{dN_1^1}{dx} = \frac{1}{l}$

$N_1^3 = \frac{x_4 - x}{l}$; $\frac{dN_1^1}{dx} = -\frac{1}{l}$

$N_2^3 = \frac{x - x_3}{l}$; $\frac{dN_1^1}{dx} = \frac{1}{l}$

$K_{11}^1 = \int\limits_l \frac{\partial N_1^1(x)}{\partial x}\frac{\partial N_1^1(x)}{\partial x}\,dx = \int\limits_l \frac{1}{l^2}\,dx = 3$

$K_{12}^1 = \int\limits_l \frac{\partial N_1^1(x)}{\partial x}\frac{\partial N_2^1(x)}{\partial x}\,dx = \int\limits_l \frac{1}{l^2}\,dx = -3$

$K_{22}^1 = \int\limits_l \frac{\partial N_2^1(x)}{\partial x}\frac{\partial N_2^1(x)}{\partial x}\,dx = \int\limits_l \frac{1}{l^2}\,dx = 3$

$K_{11}^1 = K_{11}^2 = K_{11}^3 = 3$

$K_{12}^1 = K_{12}^2 = K_{12}^3 = K_{21}^1 = K_{21}^2 = K_{21}^3 = -3$

$K_{22}^1 = K_{22}^2 = K_{22}^3 = 3$

$f_1^1 = \int\limits_0^{1/3} N_1^1(x) f(x)\, dx = \int\limits_0^{1/3} \frac{x_2 - x}{l^e} sin(x)\, dx = 0.018$

$f_2^1 = \int\limits_0^{1/3} N_2^1(x) f(x)\, dx = 0.037$

$f_1^2 = \int\limits_{1/3}^{2/3} N_1^2(x) f(x)\, dx = 0.071$

$f_2^2 = \int\limits_{1/3}^{2/3} N_2^2(x) f(x)\, dx = 0.088$

$f_1^3 = \int\limits_{2/3}^{1} N_1^3(x) f(x)\, dx = 0.117$

$f_2^3 = \int\limits_{2/3}^{1} N_2^3(x) f(x)\, dx = 0.129$

$$\begin{pmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ a_1 \\ a_2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.018 + q1 \\ 0.108 \\ 0.205 \\ 0.129 + q_4 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0.108 \\ 0.205 + 9 \end{pmatrix}$$

Finally, the results are:

$a_1 = 1.059$

$a_2 = 2.051$

$q_1 = -1.007$

$q_4 = -0.820$

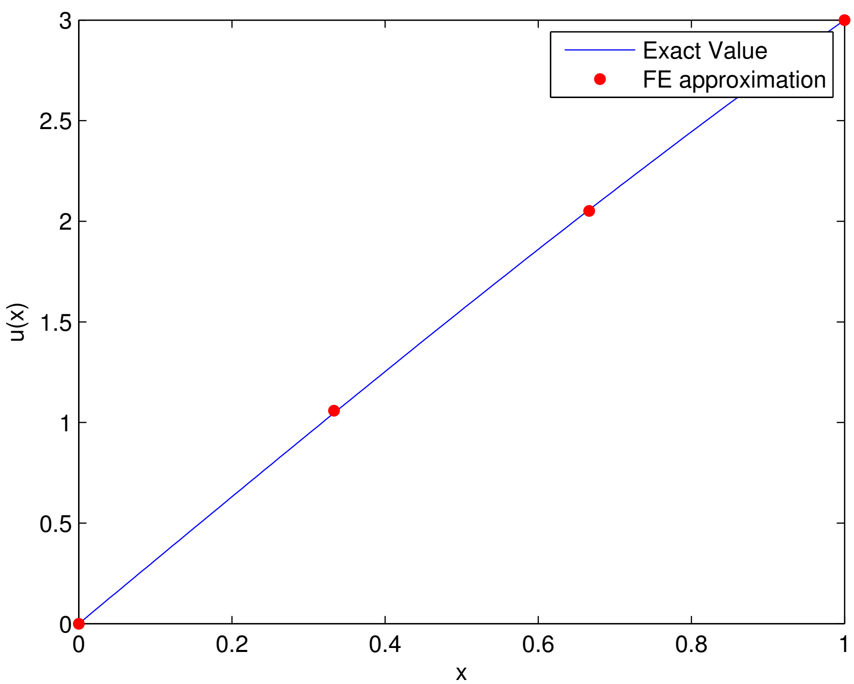

Figure 1: Comparison of exact and FE approximated values