Master on Numerical Methods in Engineering

Finite Elements
Homework 1

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PROBLEM

Consider the following differential equation

$$-u'' = f \text{ in } [0,1]$$

with the boundary conditions $u(0) = 0$ and $u(1) = \alpha$.

The Finite Element discretization is a 2-noded linear mesh given by the nodes $x_i = ih$ for $i = 0, 1, ..., n$ and $h = 1/n$.

1. Find the weak form of the problem. Describe the FE approximation u^h.
2. Describe the linear system of equations to be solved.
3. Compute de FE approximation u^h for $n = 3$, $f(x) = sin(x)$ and $\alpha = 3$. Compare it with the exact solution, $u(x) = sin(x) + [3 - sin(1)]x$.

SOLUTION

The strong form of the problem is given by:

$A(u) = \frac{d^2u}{dx^2} + Q = 0$ in $[0,1]$

$B(u) = u - \overline{u} = 0$ in $x = 0$ and $x = 1$

We don't have any Neumann boundary condition, therefore the integral form of the problem is:

$\int_0^1 \omega[\frac{d^2u}{dx^2} + Q]dx = \int_0^1 \omega\frac{d^2u}{dx^2}dx + \int_0^1 \omega Q(x)dx = 0$

Integrating by parts the first term of the equation we find the weak form of the problem

$-\int_0^1 \frac{d\omega}{dx}\frac{du}{dx}dx + \omega\frac{du}{dx}\Big|_0^1 + \int_0^1 \omega Q(x)dx = 0$

We can approximate the function $u(x)$ as $u^h(x) = N_i(x)u_i$, call $\frac{du}{dx}$ the reaction force/flux q, use the Galerking method choosing $\omega(x) = N_i(x)$ and rearrange the terms of the equation. Then, the weak form of the problem will look like this:

$\int_0^1 \frac{dN_i}{dx} \sum_{j=1}^n (\frac{dN_j}{dx} u_j) dx = N_i q\Big|_0^1 + \int_0^1 N_i Q(x) dx$

This expression represents a linear system of equation of the form $K_{ij} u_j = f_i$.

K_{ij} is a square symmetric matrix of size n x n. Where n is the number of nodes in the discretization.

$K_{ij} = \int_0^1 \frac{dN_i}{dx} \frac{dN_j}{dx} dx$

u_j is a vector of size n with the values of the unknown in the nodes.

f_i is a vector of size n with the independent terms of the equation.

$f_i = N_i q\Big|_0^1 + \int_0^1 N_i Q(x) dx$

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & \dots & K_{1n} \\ K_{21} & K_{22} & K_{23} & \dots & K_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ K_{n1} & K_{n2} & K_{n3} & \dots & K_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \dots \\ f_n \end{bmatrix}$$

For a 3 finite element discretization of the domain we can find the system of equation to solve by assembling the local components of K and f of each element.

For each element, the K matrix will be

$K^e = \frac{1}{1/3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

And the f vector will depend on the integration over the length of the element of the function $Q(x) = sin(x)$ plus the boundary flux/reaction q when pertinent.

Knowing that locally the shape function has the form of:

$N_1^e = \frac{x_2 - x}{l^e}$

$N_2^e = \frac{x - x_1}{l^e}$

We can find the 6 shape functions N_i for each node of the 3 elements.

$N_1^1 = 3(1/3 - x)$
$N_2^1 = 3(x - 0)$
$N_1^2 = 3(2/3 - x)$
$N_2^2 = 3(x - 1/3)$
$N_1^3 = 3(1 - x)$
$N_2^3 = 3(x - 2/3)$

f will take a value in each node of the elements of:

$f_1^1 = \int_0^{1/3} (-3x + 1) sin(x) dx + q_1 = 0.018 + q_1$
$f_2^1 = \int_0^{1/3} (3x) sin(x) dx = 0.037$

$f_1^2 = \int_{1/3}^{2/3}(-3x+2)sin(x)dx = 0.071$
$f_2^2 = \int_{1/3}^{2/3}(3x-1)sin(x)dx = 0.088$
$f_1^3 = \int_{2/3}^{1}(-3x+3)sin(x)dx = 0.117$
$f_2^3 = \int_{2/3}^{1}(3x-2)sin(x)dx + q_4 = 0.129 + q_4$

Assembled, the general system of equation will take the form of:

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 \\ K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & 0 \\ 0 & K_{21}^2 & K_{22}^2 + K_{11}^3 & K_{12}^3 \\ 0 & 0 & K_{21}^3 & K_{22}^3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ f_2^3 \end{bmatrix}$$

Replacing the values of K and f previously found and introducing the Dirichlet boundary conditions we get ($\alpha = 3$):

$$3\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ u_3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.018 + q_1 \\ 0.108 \\ 0.204 \\ 0.129 + q_4 \end{bmatrix}$$

As u_1 and u_4 are known values we can reduce the system to:

$$3\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.108 \\ 0.204 + 9 \end{bmatrix}$$

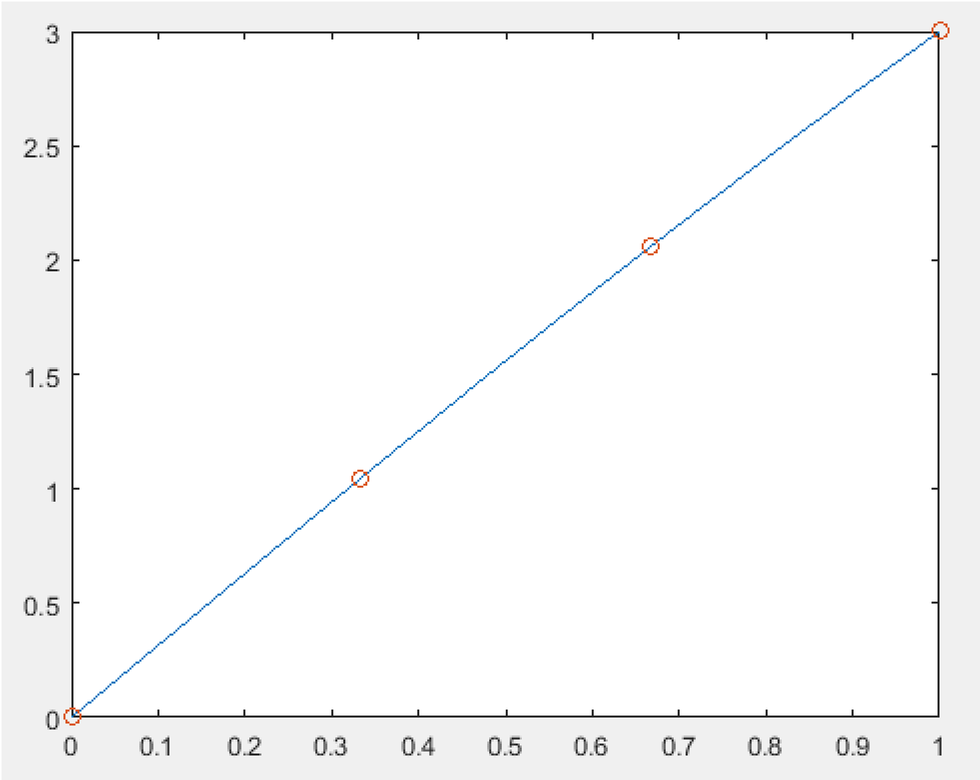
Solving the system we find that $u_2 = 1.046$ and $u_3 = 2.057$

Now, with all the node values known we can compute the reactions/fluxes q_1 and q_4

$-3(1.059) = 0.018 + q_1 \rightarrow q_1 = -3.158$
$-3(2.051) + 3(3) = 0.129 + q_4 \rightarrow q_4 = 2.698$

We also can compute the values of u in the same points but using the real governing equation $u(x) = sin(x) + [3 - sin(1)]x$

Node	u_{fem}	u_{exact}
1	0	0
2	1.046	1.046
3	2.057	2.057
4	3	3



Graphic: The blue line represents the exact solution while the red points represent the values found with the finite element method.

The approximation results were the same as the ones find by the exact expression, at least for the precision used for the calculations. Anyway, as the exact expression is not a linear function, there is a difference between values evaluated between points, because u^h is an interpolation of the values found by FEM.