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Consider the following differential equation

$$-u'' = f\ in\ ]0,1[$$

with the boundary conditions $u(0)$=0, $u(1)$=α.

The Finite Element discretization is a 2-noded linear mesh given by the nodes $x_i = ih$ for $i = 0,1,\dots,n\ and\ h = 1/n$.

1. Find the weak form of the problem. Describe the FE approximation u^h.
2. Describe the linear system of equations to be solved.
3. Compute the FE approximation u^h for $n = 3, f(x) = \sin x\,, and\ \alpha = 3$. Compare it with the exact solution $u(x) = \sin x + (3 - \sin 1)x$.

Solution

1. Differential equation governing the problem:

$$A(u) = \frac{d^2u}{x^2} + f(x) = 0\ in\ \Omega =]0,1[$$

Boundary condition equations: $B(u): \begin{cases} u(0) = 0 \\ u(1) = 0 \end{cases}\ on\ \Gamma_u$

For 1D equation takes the form for arbitrary function W(x)

$\int_0^1 f(x)W(x)dx = -\int_0^1 u''(x)W(x)dx,$

Integrating by parts $-\int_0^1 u''(x)W(x)dx = -u'(x)W(x)|_0^1 + \int_0^1 u'(x)W'(x)dx = \int_0^1 u'(x)W'(x)dx - [u'(x)W(x)]_1 + [u'(x)W(x)]_0$

Receive weak form $\int_0^1 f(x)W(x)dx = \int_0^1 u'(x)W'(x)dx + [qW(x)]_1 - [qW(x)]_0$ where $q = -ku'(x) = -u'(x)$ as in our case $k = 1$. Here q is the heat flux.

Approximating numerical solution with a linear combination of function:

$$u \cong u^h = \sum_{i=1}^{n} N_i(x)u_i$$

Substitute the approximation function into the weak form:

$$\int_0^1 \frac{du^h}{dx}\frac{dW_i}{dx}dx = \int_0^1 fW_i(x)dx - [qW(x)]_1 + [qW(x)]_0$$

$$\int_0^1 \frac{d\sum_{j=1}^{n} N_j(x)u_j}{dx}\frac{dW_i}{dx}dx = \int_0^1 fW_i(x)dx - [qW_i(x)]_1 + [qW_i(x)]_0$$

Using Galerkin method, choose the weighted function $W_i(x) \equiv N_i(x)$.
In this case the weak form takes the view:

$$\int_0^1 \frac{d\sum_{j=1}^{n} N_j(x)u_j}{dx}\frac{dN_i}{dx}dx = \int_0^1 fN_i(x)dx - [qN_i(x)]_1 + [qN_i(x)]_0$$

2. From the last weak form we can obtain global system of equations:

$$\sum_{j=1}^{n}\left(u_j \int_0^1 \frac{dN_i}{dx}\frac{dN_j}{dx}\, dx\right) = \int_0^1 fN_i(x)dx - [qN_i(x)]_1 + [qN_i(x)]_0$$

The components u_j could be found by solving the system of n equations $\boldsymbol{Ku} = \boldsymbol{f}$, where $\boldsymbol{K}$ is such matrix as $K_{ij} = \int_0^1 \frac{dN_i}{dx}\frac{dN_j}{dx}\, dx$, and $\boldsymbol{f}$ is such vector that $f_i = \int_0^1 fN_i(x)dx - [qN_i(x)]_1 + [qN_i(x)]_0$.

Discretization of the domain for a mesh of two-noded elements:

$$\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & \cdots & 0 & \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & & & \\ \vdots & & \ddots & \vdots & \\ 0 & & \cdots & K_{22}^{(n-1)} + K_{11}^{(n)} & K_{12}^{(n)} \\ & & & K_{21}^{(n)} & K_{22}^{(n)} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} f_1^{(1)} + q_0 \\ f_2^{(1)} + f_1^{(2)} \\ \vdots \\ f_2^{(n-1)} + f_1^{(n)} \\ f_2^{(n)} - q_l \end{bmatrix}$$

Where **K** – the global stiffness matrix such as

$$K_{ij}^{(e)} = \int_{l^{(e)}} \frac{dN_i^{(e)}}{dx}\frac{dN_j^{(e)}}{dx}\, dx = (-1)^{i+j}\left(\frac{1}{l}\right)^{(e)}$$

And components of the global equivalent nodal flux vector **f:**

$$f_i^{(e)} = \int_{l^{(e)}} fN_i^{(e)}(x)dx$$

$N_i^{(e)}$ is defined as

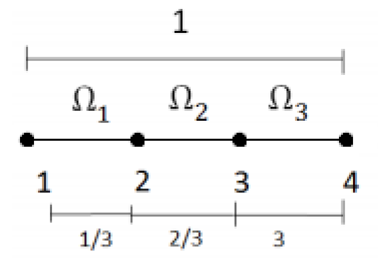
$$N_1^{(e)}(x) = \frac{x_2^{(e)} - x}{l^{(e)}},\ N_2^{(e)}(x) = \frac{x - x_1^{(e)}}{l^{(e)}}$$

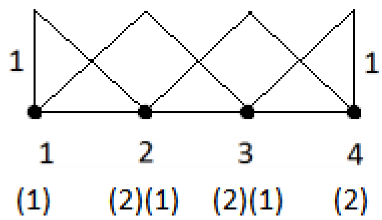
$$N_i^{(e)}(x) = \begin{cases} N_i^{(e)}(x_i) = 1 \\ N_i^{(e)}(x_j) = 0 \end{cases}$$

3. Computing the FE approximation u^h for $n = 3, f(x) = \sin x\, , and\ \alpha = 3$

As $n = 3 \Rightarrow h = \frac{1}{n} = 1/3$. Then as $x_i = ih$, $x_0 = 0 * \frac{1}{3} = 0, x_1 = 1 * \frac{1}{3} = 1/3, x_2 = 2 * \frac{1}{3} = 2/3, x_3 = 3 * \frac{1}{3} = 1.$

$$l^{(1)} = l^{(2)} = l^{(3)} = 1/3$$





System of equations:

$$\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & 0 \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} + K_{11}^{(3)} & K_{12}^{(3)} \\ 0 & 0 & K_{12}^{(3)} & K_{22}^{(3)} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1^{(1)} + q_0 \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ f_2^{(3)} + q_l \end{bmatrix}$$

$$K_{11}^{(1)} = K_{11}^{(2)} = K_{11}^{(3)} = K_{22}^{(1)} = K_{22}^{(2)} = K_{22}^{(3)} = (-1)^{i+j}\left(\frac{1}{l}\right)^{(e)} = \frac{1}{\frac{1}{3}} = 3$$

$$K_{12}^{(1)} = K_{12}^{(2)} = K_{12}^{(3)} = K_{21}^{(1)} = K_{21}^{(2)} = K_{21}^{(3)} = (-1)^{i+j}\left(\frac{1}{l}\right)^{(e)} = -\frac{1}{\frac{1}{3}} = -3$$

$$f_1^{(1)} = \int_0^{1/3} N_1^{(1)}(x)\sin x\ dx = \int_0^{1/3} \frac{x_2^{(1)} - x}{l^{(1)}}\sin x\ dx = \int_0^{1/3} \frac{1/3 - x}{1/3}\sin x\ dx = \int_0^{1/3} \sin x\ dx$$

$$- 3\int_0^{1/3} x\sin x\ dx = -\cos x\,|_0^{1/3} - 3(-x\cos x + \sin x)\,|_0^{\frac{1}{3}}$$

$$= -\cos\frac{1}{3} + 1 + \cos\frac{1}{3} - 3\sin\frac{1}{3} = 1 - 3\sin\frac{1}{3} = 0.0184$$

$$f_1^{(2)} = \int_{1/3}^{2/3} N_1^{(2)}(x)\sin x\ dx = \int_{1/3}^{2/3} \frac{x_2^{(2)} - x}{l^{(2)}}\sin x\ dx = \int_{1/3}^{2/3} \frac{2/3 - x}{1/3}\sin x\ dx$$

$$= 2\int_{1/3}^{2/3} \sin x\ dx - 3\int_{1/3}^{2/3} x\sin x\ dx$$

$$= -2\cos x\,|_{1/3}^{2/3} - 3(-x\cos x + \sin x)\,|_{1/3}^{2/3} = \cos\frac{1}{3} + 3\left(\sin\frac{1}{3} - \sin\frac{2}{3}\right)$$

$$= 0.0714$$

$$f_1^{(3)} = \int_{2/3}^{1} N_1^{(3)}(x)\sin x\ dx = \int_{2/3}^{1} \frac{x_2^{(3)} - x}{l^{(3)}}\sin x\ dx = \int_{2/3}^{1} \frac{1 - x}{1/3}\sin x\ dx$$

$$= 3\int_{2/3}^{1} \sin x\ dx - 3\int_{2/3}^{1} x\sin x\ dx = -3\cos x\ |_{2/3}^{1} - 3(-x\cos x + \sin x)\ |_{2/3}^{1}$$

$$= \cos\frac{2}{3} + 3\left(\sin\frac{2}{3} - \sin 1\right) = 0.1166$$

$$f_2^{(1)} = \int_{0}^{1/3} N_2^{(1)}(x)\sin x\ dx = \int_{0}^{1/3} \frac{x - x_1^{(1)}}{l^{(1)}}\sin x\ dx = \int_{0}^{1/3} \frac{x - 0}{1/3}\sin x\ dx$$

$$= 3\int_{0}^{1/3} x\sin x\ dx = 3(-x\cos x + \sin x)\ |_{0}^{\frac{1}{3}} = -\cos\frac{1}{3} + 3\sin\frac{1}{3} = 0.0366$$

$$f_2^{(2)} = \int_{1/3}^{2/3} N_2^{(2)}(x)\sin x\ dx = \int_{1/3}^{2/3} \frac{x - x_1^{(2)}}{l^{(2)}}\sin x\ dx = \int_{1/3}^{2/3} \frac{x - 1/3}{1/3}\sin x\ dx$$

$$= 3\int_{1/3}^{2/3} x\sin x\ dx - \int_{1/3}^{2/3} \sin x\ dx = 3(-x\cos x + \sin x)\ |_{1/3}^{2/3} + \cos x\ |_{1/3}^{2/3}$$

$$= -\cos\frac{2}{3} + 3\sin\frac{2}{3} - 3\sin\frac{1}{3} = 0.0876$$

$$f_2^{(3)} = \int_{2/3}^{1} N_2^{(3)}(x)\sin x\ dx = \int_{2/3}^{1} \frac{x - x_1^{(3)}}{l^{(3)}}\sin x\ dx = \int_{2/3}^{1} \frac{x - 2/3}{1/3}\sin x\ dx$$

$$= 3\int_{2/3}^{1} x\sin x\ dx - 2\int_{2/3}^{1} \sin x\ dx = 3(-x\cos x + \sin x)\ |_{2/3}^{1} + 2\cos x\ |_{2/3}^{1}$$

$$= -\cos 1 + 3\left(\sin 1 - \sin\frac{2}{3}\right) = 0.1290$$

Because of boundary conditions $u_1 = 0, u_4 = \alpha = 3$. The system of equations takes the view:

$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ u_3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.0184 + q_0 \\ 0.1080 \\ 0.2042 \\ 0.1290 + q_l \end{bmatrix}$$

Solving the system $\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.1080 \\ 9.2042 \end{bmatrix}$, receive $u_2 = 1.0467, u_3 = 2.0573.$

Now we can calculate the flux on the boundaries:

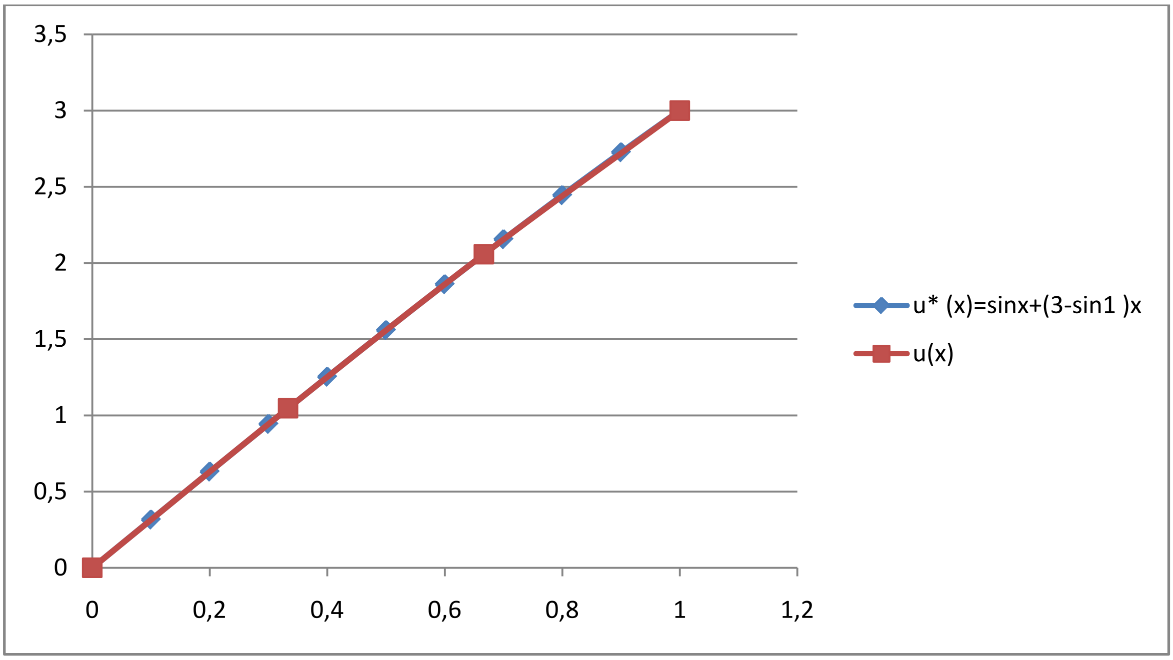
$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1.0467 \\ 2.0573 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.0184 + q_0 \\ 0.1080 \\ 0.2042 \\ 0.1290 + q_l \end{bmatrix}$$

$$-3 * 1.0467 = 0.0184 + q_0 \Rightarrow q_0 = -3.1585$$
$$2.0573 * 3 + 3 * 3 = 0.1290 + q_l \Rightarrow q_l = -2.6991$$

The fluxes are negative because their directions are opposite to that we assumed.

Let us compare FEM solution u with the exact solution $u^*(x) = \sin x + (3 - \sin 1)x$.

	$u^*(x) = \sin x + (3 - \sin 1)x$	$u(x)$
0	0	0
1/3	1.0467	1.0467
2/3	2.0574	2.0573
1	3	3



As we can see from the table and the graph, the approximate solution is equal to the exact solution. Two displacements u_2 and u_3 obtain the difference between the exact solution and the FEM is 0 and 0.0001 respectively. The approximations converge with a minimum error of 0.01% in the case of u_3 and 0% in u_2 . The exact solution can be approximated by a linear function (FE 2 node mesh with three elements).