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Tasks

Solve the following problem with the MATLAB PDE Toolbox:

$$u_t - \Delta u = f \quad \text{in } \Omega = [0, 1]^2,$$

where the source term is given by,

$$f(x, y, t) = -3e^{-3t}.$$

We consider an initial condition at $t = 0$:

$$u(x, y, t = 0) = x^2 + xy - y^2 + 1,$$

and the following boundary conditions

$$\begin{aligned} u_n(x = 0, y, t) &= -y, \\ u_n(x = 1, y, t) &= 2 + y, \\ u(x, y = 0, t) &= x^2 + e^{-3t}, \\ u_n(x, y = 1, t) &= x - 2, \end{aligned}$$

where $u_n \equiv \partial u / \partial n$.

The analytical solution of this problem is given by the following expression,

$$u(x, y, t) = x^2 + xy - y^2 + e^{-3t}.$$

1. Consider $t_{end} = 10$, solve the problem, and refine the initial mesh up to 4 times. Verify that the theoretical convergence order holds.
2. How is the solution affected when we modify the final time?
3. We are interested in obtaining the solution at time $t_{end} = 50$. Find a more efficient manner to solve this problem. You do not need to prove the equivalence mathematically, but you need to provide numerical evidence of the new method.

Solution

The problem consists in solve a partial differential equation using Toolbox of Matlab. The equation given is a *Parabolic* having the following form:

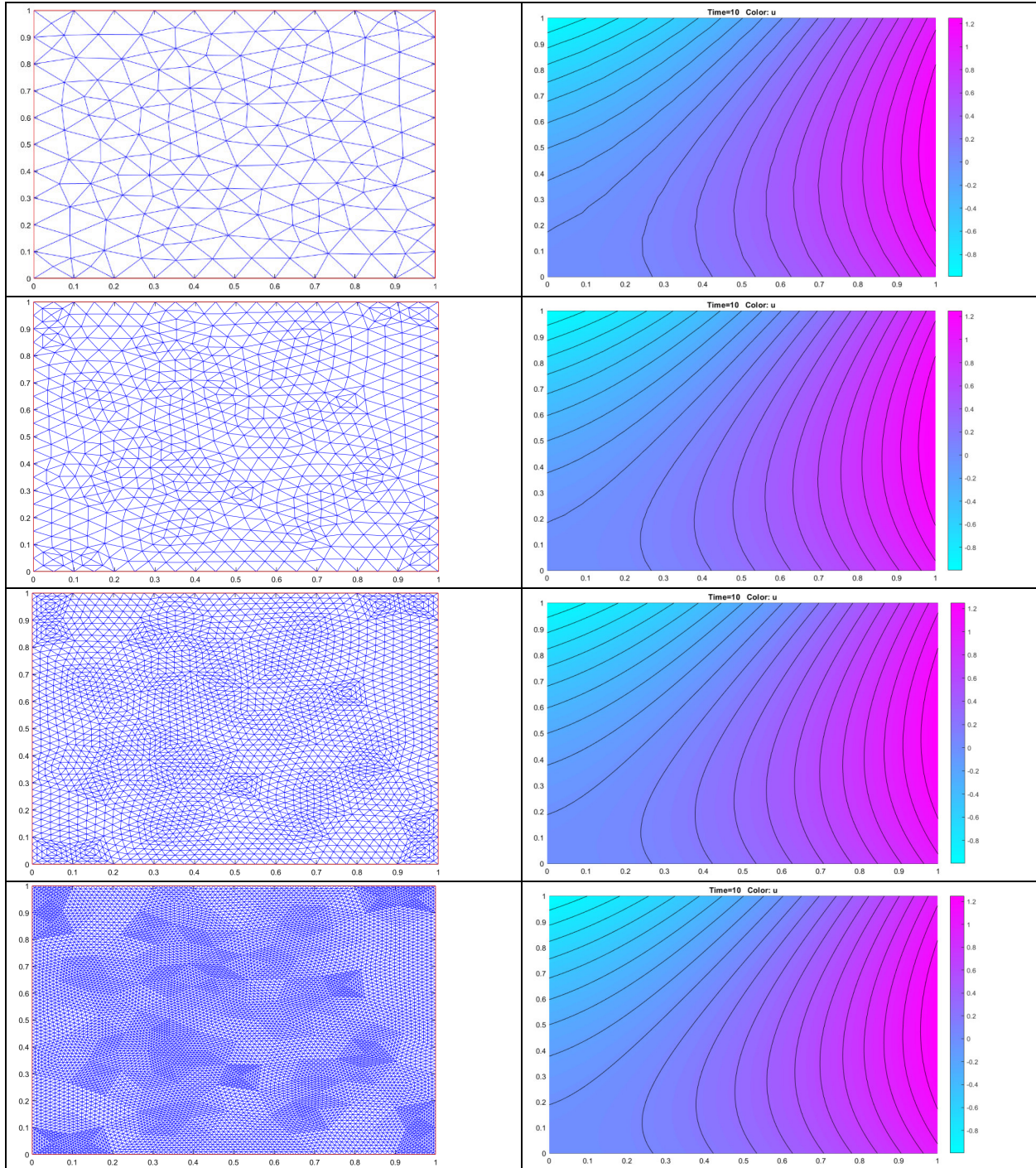
$$d \frac{\partial u}{\partial t} - \nabla \cdot (c \nabla u) + au = f$$

Making a comparison with our equation, we found:

$\frac{\partial u}{\partial t} = u_t \rightarrow d = 1$	$\Delta = \nabla \cdot \nabla \rightarrow c = 1$	$au = 0; f = -3e^{-3t}$
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The domain is a square of 1.0 m side. Also, we have three Neumann conditions and 1 Dirichlet condition.

1) Using MATLAB and pdetool



With the purpose of make a convergency analysis, the data of each mesh and the solutions are export in order to compute the error for each mesh size. The following table shows the value of error for 5 different sizes of meshes, with 1x the thicker mesh and 5x the finer mesh. The height h was computed as an average using the following expression:

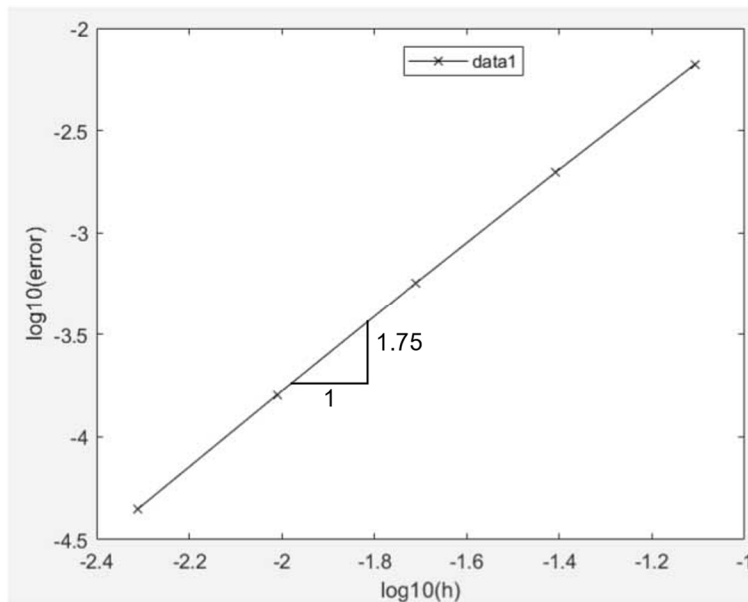
$$h = \sqrt{\frac{2 \times A_{total}}{n_e}}$$

Donde

A_{total} = total area of the domain 1 m^2

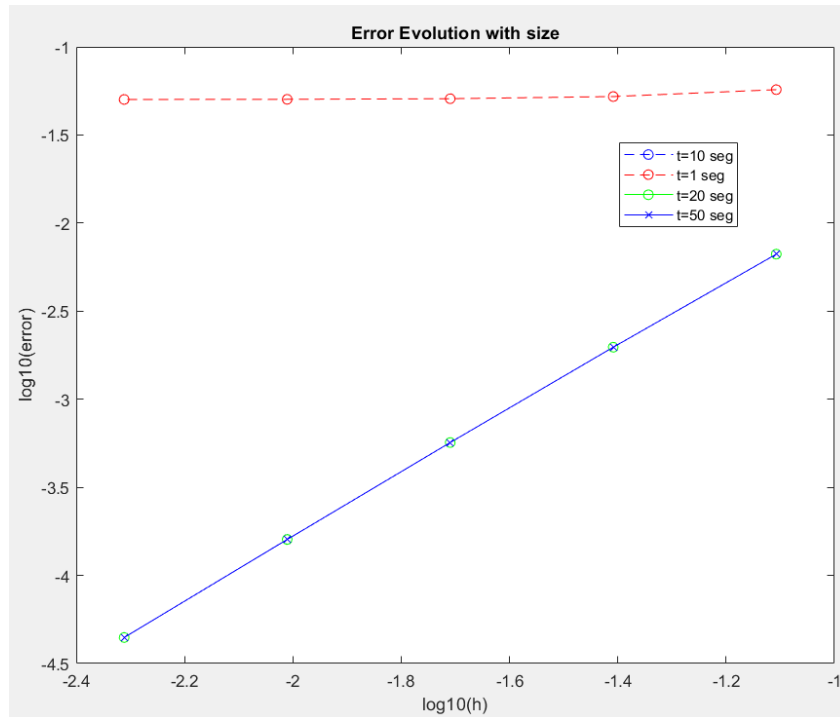
n_e = number of total elements in the domain (taken from pde-tool)

Mesh	n_e	h	Error
1x	328	0.078086880944303	0.00666213527140958
2x	1312	0.0390434404721515	0.00197333229663843
3x	5248	0.0195217202360758	0.000567874536162361
4x	20992	0.00976086011803788	0.000160036444216694
5x	83968	0.00488043005901894	0.0000444234348354611



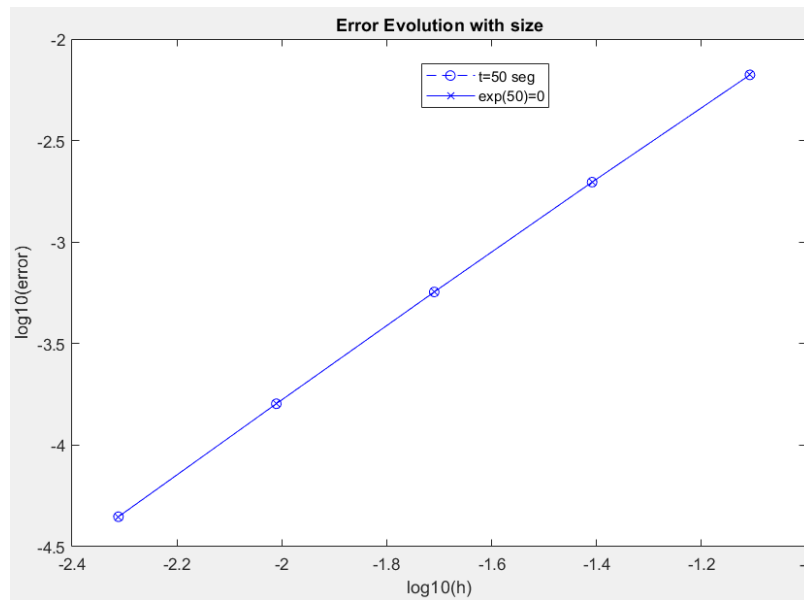
It is shown that the slope of the plot is near to the ratio 1:2 which indicates a good solution of the equation.

2) if we change the final time to 1 seg, the results are:



The error increases considerably if we compare with the solution for a time $t=10$ seg. At the same time, for time higher than 10 seg the solutions seem to be the same or very the same, this indicates to get a good solution it is just necessary a final time of 10 seg.

3) In order to introduce the approximation $e=0$ we neglected the term of the function f and the last term in the boundary condition for $y=0$.



The solution is the same as if we just consider $t=50$ seg, but the time of the calculations can be reduced introducing the approximation $e \approx 0$.