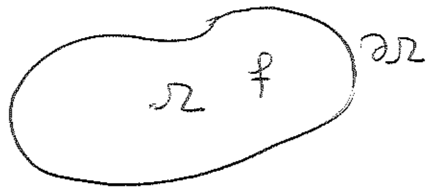
2. COUPLING IN SPACE OF HOMOGENEOUS PROBLEMS: DOMAIN DECOMPOSITION METHODS

1. INTRODUCTION

We wish to solve:

$$\mathcal{L}u = f \text{ in } \Omega$$
$$u = \bar{u} \text{ on } \partial\Omega$$



For example, $\mathcal{L}u = -\nabla \cdot k \nabla u$.

Procedure

a) Split Ω into s subdomains $\Omega_1, \ldots, \Omega_S$, i.e., $\Omega = \bigcup_{i=1}^{S} \Omega_i$

b) Write down the transmission conditions between subdomains

c) Solve a boundary value problem within each subdomain Ω_i

d) Obtain the solution in Ω (global solution)

- Using an iterative method
- " a direct "

Motivation

Suppose that when the problem is discretized, each subdomain has n degrees of freedom (dof).

Cost of solving a problem $\propto (\text{dof})^p$, $p > 1$.

$(sn)^p > s n^p \Rightarrow$ less cost to solve each system

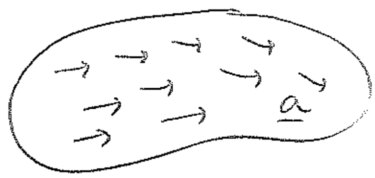
$(sn)^2 > s n^2 \Rightarrow$ less components to be stored in the matrices of the discrete problem

- Parallelization: very often, the local problems on Ω_i, $i = 1, \ldots, s$, can be solved in parallel
- Physics: DD allow to couple
  - Different physics
  - Domains in relative motion.

2. CONVECTION-DIFFUSION EQUATION

Continuous problem

$$-k\Delta u + a\cdot\nabla u + su = f \quad \text{in } \Omega$$
$$u = \bar{u} \quad \text{on } \partial\Omega$$



$k>0, \quad \nabla\cdot a = 0, \quad s \geq 0$

Weak form:

$$\int_\Omega \delta u(-k\Delta u + a\cdot\nabla u + su) = \int_\Omega \delta u f \quad \forall \delta u$$

$$\int_\Omega k\nabla\delta u\cdot\nabla u + \int_\Omega \delta u\, a\cdot\nabla u + \int_\Omega \delta u\, s u = \int_\Omega \delta u f \quad \forall \delta u \mid \delta u = 0 \text{ on } \partial\Omega$$

$$V(\bar{u}) := \{ v \in H^1(\Omega) \mid v = \bar{u} \text{ on } \partial\Omega \}$$

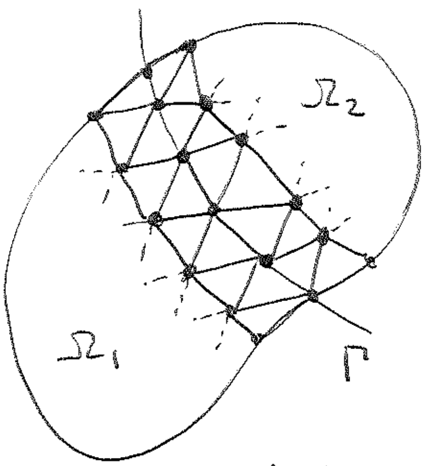
Find $u \in V(\bar{u})$ such that

$$B(u,\delta u) = L(\delta u) \quad \forall \delta u \mid \delta u \in V(0)$$

$$B(u,\delta u) = \int_\Omega k\nabla\delta u\cdot\nabla u + \int_\Omega \delta u\, a\cdot\nabla u + \int_\Omega \delta u\, s u$$

$$L(\delta u) = \int_\Omega \delta u f$$

Finite element approximation



$\Omega = \bigcup K$, f.e. partition

$u \approx u_h, \quad u_h|_K \in P_p(K)$

$$\left.\begin{aligned} &u_h|_K \text{ polynomial} \\ &u_h \in H^1(\Omega) \end{aligned}\right\} \Rightarrow u_h \in C^0(\Omega)$$

$$u_h(x) = \sum_a N^a(x)\, U^a, \quad N^a(x)\big|_K \in P_p(K)$$

Consider nodal degrees of freedom:

$$N^a(x^b) = \delta^{ab}, \quad U^a = u_h(x^a)$$

$$V_h^p(\bar{u}) = \{ v \in V(\bar{u}) \mid v|_K \in P_p(K) \}$$

Discret problem: find $u_h \in V_h(\bar{u})$ such that

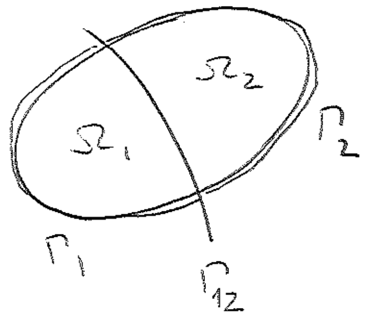
$$B(u_h, \delta u_h) = L(\delta u_h) \quad \forall \, \delta u_h \in V_h(0)$$

$$\Leftrightarrow \quad \sum_{a,b} \delta U^b \underbrace{B(N^a, N^b)}_{A^{ba}} U^a = \sum_b \delta U^b \underbrace{L(N^b)}_{F^b} \quad \forall \, \delta U$$

$$\therefore \quad \boxed{AU = F} \quad \text{(with or without BC's)}$$

3. NON-OVERLAPPING DOMAIN DECOMPOSITION METHODS

3.1. GEOMETRIC VERSION (CONTINUOUS PROBLEM)



$$\bar{\Omega} = \overline{\Omega_1 \cup \Omega_2}$$

$$\Gamma_{12} = \bar{\Omega}_1 \cap \bar{\Omega}_2$$

$$\Gamma_i = \partial\Omega \cap \bar{\Omega}_i$$

If $u_i := u|_{\Omega_i}$, there holds:

$$\mathcal{L} u_i = f \quad \text{in } \Omega_i \quad , i = 1,2$$

$$u_i = \bar{u}|_{\Gamma_i} \quad \text{on } \Gamma_i \quad , i = 1,2$$

$$\boxed{\begin{aligned} u_1 &= u_2 \\ k_1 \frac{\partial u_1}{\partial n} &= k_2 \frac{\partial u_2}{\partial n} \end{aligned}} \quad \text{on } \Gamma_{12} \text{ for a fixed } n$$

Direct method: Steklov – Poincaré operator

Let $u_i = u_i^0 + \tilde{u}_i$, $i = 1,2$, with:

$$\left. \begin{aligned} \mathcal{L} u_i^0 &= f \quad \text{in } \Omega_i \\ u_i^0 &= 0 \quad \text{on } \Gamma_i \\ u_i^0 &= 0 \quad \text{on } \Gamma_{12} \end{aligned} \right\} \qquad \left. \begin{aligned} \mathcal{L} \tilde{u}_i &= 0 \quad \text{in } \Omega_i \\ \tilde{u}_i &= \bar{u}|_{\Gamma_i} \quad \text{on } \Gamma_i \\ \tilde{u}_i &= \varphi \quad \text{on } \Gamma_{12} \end{aligned} \right\}$$

Problem: obtain φ such that $u_i = u_i^0 + \tilde{u}_i$ is $u|_{\Omega_i}$. We must ensure that:

$$k_1 \frac{\partial u_1}{\partial n} = k_2 \frac{\partial u_2}{\partial n} \iff k_1 \frac{\partial \tilde{u}_1}{\partial n} - k_2 \frac{\partial \tilde{u}_2}{\partial n} = -k_1 \frac{\partial u_1^0}{\partial n} + k_2 \frac{\partial u_2^0}{\partial n}$$

Define:

$$\mathcal{S} : H^{1/2}(\Gamma_{12}) \longrightarrow H^{-1/2}(\Gamma_{12})$$

$$\varphi \longmapsto k_1 \frac{\partial \tilde{u}_1}{\partial n} - k_2 \frac{\partial \tilde{u}_2}{\partial n}$$

$$\mathcal{G} = -k_1 \frac{\partial u_1^0}{\partial n} + k_2 \frac{\partial u_2^0}{\partial n} \in H^{-1/2}(\Gamma_{12})$$

Problem: find $\varphi \in H^{1/2}(\Gamma_{12})$ such that

$$\mathcal{S}\varphi = \mathcal{G}$$

<u>Def</u>: $\mathcal{S}$: Steklov–Poincaré operator.

<u>Iteration-by-subdomains: Jacobi and Gauss–Seidel methods</u>

<u>Problem in Ω_1:</u>	<u>Problem in Ω_2</u>
$\mathcal{L} u_1^{(k)} = f$	$\mathcal{L} u_2^{(k)} = f$
$u_1^{(k)} = \bar{u}\|_{\Gamma_1}$	$u_2^{(k)} = \bar{u}\|_{\Gamma_2}$
$k_1 \frac{\partial u_1^{(k)}}{\partial n} = k_2 \frac{\partial u_2^{(k-1)}}{\partial n}$ on Γ_{12}	$u_2^{(k)} = u_1^{(\ell)}$ on Γ_{12}

Dirichlet – Neumann (DN) coupling.

$\ell = k-1$. Jacobi scheme (parallel)

$\ell = k$ Gauss–Seidel scheme (sequential)

<u>Remark</u> At the discrete level, Dirichlet conditions can be prescribed in a strong way or in a weak way, using the so-called "mortar elements". Neumann conditions need to be prescribed weakly.

Other methods

- Neumann - Neumann algorithm:

$$\left.\begin{aligned} \mathcal{L}u_1^{(k+1)} &= f && \text{in } \Omega_1 \\ u_1^{(k+1)} &= \varphi^{(k)} && \text{on } \Gamma_{12} \\ u_1^{(k+1)} &= \bar{u}|_{\Gamma_1} && \text{on } \Gamma_1 \end{aligned}\right| \quad \left.\begin{aligned} \mathcal{L}u_2^{(k+1)} &= f && \text{in } \Omega_1 \\ u_2^{(k+1)} &= \varphi^{(k)} && \text{on } \Gamma_{12} \\ u_2^{(k+1)} &= \bar{u}|_{\Gamma_2} && \text{on } \Gamma_2 \end{aligned}\right\}$$

$$\left.\begin{aligned} \mathcal{L}\psi_1^{(k+1)} &= 0 \\ k_1 \frac{\partial \psi_1^{(k+1)}}{\partial n} &= k_1 \frac{\partial u_1^{(k+1)}}{\partial n} - k_2 \frac{\partial u_2^{(k+1)}}{\partial n} \\ \psi_{\Gamma_1}^{(k+1)} &= 0 \end{aligned}\right\rangle \quad \left.\begin{aligned} \mathcal{L}\psi_2^{(k+1)} &= 0 \\ k_2 \frac{\partial \psi_2^{(k+1)}}{\partial n} &= k_1 \frac{\partial u_1^{(k+1)}}{\partial n} - k_2 \frac{\partial u_2^{(k+1)}}{\partial n} \\ \psi_2^{(k+1)} &= 0 \end{aligned}\right\rangle$$

$$\varphi^{(k+1)} = \varphi^{(k)} - \theta\left(\sigma_1 \psi_1^{(k+1)}\Big|_{\Gamma_{12}} - \sigma_2 \psi_2^{(k+1)}\Big|_{\Gamma_{12}}\right)$$

Local problems can be solved in parallel.

- Robin - Robin algorithm:

<u>Problem in Ω_1</u>

$$\begin{aligned} \mathcal{L}u_1^{(k)} &= f \quad \text{in } \Omega_1 \\ u_1^{(k)} &= \bar{u}|_{\Gamma_1} \quad \text{on } \Gamma_1 \\ k_1 \frac{\partial u_1^{(k)}}{\partial n} + \gamma_1 u_1^{(k)} &= k_2 \frac{\partial u_2^{(k-1)}}{\partial n} + \gamma_1 u_2^{(k-1)} \end{aligned}$$

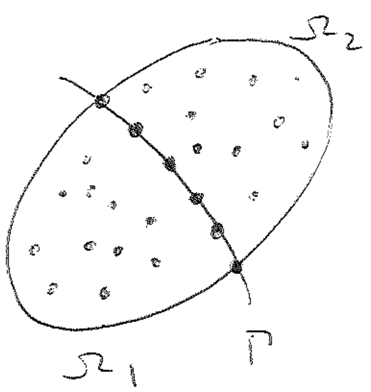
<u>Problem in Ω_2</u>

$$\begin{aligned} \mathcal{L}u_2^{(k)} &= f \quad \text{in } \Omega_1 \\ u_2^{(k)} &= \bar{u}|_{\Gamma_2} \quad \text{on } \Gamma_2 \\ k_2 \frac{\partial u_2^{(k)}}{\partial n} + \gamma_2 u_2^{(k)} &= k_1 \frac{\partial u_1^{(\ell)}}{\partial n} + \gamma_2 u_1^{(\ell)} \end{aligned}$$

with $\gamma_1 + \gamma_2 > 0$.

$\ell = k-1$: Jacobi scheme (parallel)
$\ell = k$: Gauss - Seidel " (sequential)

3.2. ALGEBRAIC VERSION (DISCRETE PROBLEM)



Discrete problem:

$$AU = F$$

$$\begin{bmatrix} A_{11} & A_{1\Gamma} & 0 \\ A_{\Gamma 1} & A_{\Gamma\Gamma} & A_{\Gamma 2} \\ 0 & A_{2\Gamma} & A_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_\Gamma \\ U_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma \\ F_2 \end{bmatrix}$$

$$U_1 = A_{11}^{-1}\left(F_1 - A_{1\Gamma} U_\Gamma\right)$$
$$U_2 = A_{22}^{-1}\left(F_2 - A_{2\Gamma} U_\Gamma\right)$$

$$A_{\Gamma 1} A_{11}^{-1}\left(F_1 - A_{1\Gamma} U_\Gamma\right) + A_{\Gamma\Gamma} U_\Gamma + A_{\Gamma 2} A_{22}^{-1}\left(F_2 - A_{2\Gamma} U_\Gamma\right) = F_\Gamma$$

$$\underbrace{\left(A_{\Gamma\Gamma} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}\right)}_{S} U_\Gamma = \underbrace{F_\Gamma - A_{\Gamma 1} A_{11}^{-1} F_1 - A_{\Gamma 2} A_{22}^{-1} F_2}_{G}$$

$$\boxed{S U_\Gamma = G}$$

<u>Def</u>: S : Schur complement. It is exactly the discrete version of the Steklov - Poincaré operator.

<u>Remark</u> If an iterative algebraic solver is used to solve $SU_\Gamma = G$, one only needs to evaluate

$$z^{(k)} = S U_\Gamma^{(k)} = A_{\Gamma\Gamma} U_\Gamma^{(k)} - \underbrace{A_{\Gamma 1} \underbrace{A_{11}^{-1} \underbrace{A_{1\Gamma} U_\Gamma^{(k)}}_{y_1^{(k)}}}_{x_1^{(k)}}}_{z_1^{(k)}} - \underbrace{A_{\Gamma 2} \underbrace{A_{22}^{-1} \underbrace{A_{2\Gamma} U_\Gamma^{(k)}}_{y_2^{(k)}}}_{x_2^{(k)}}}_{z_2^{(k)}}$$

- Evaluate $y_i^{(k)} = A_{i\Gamma} U_\Gamma^{(k)}$
- Solve $A_{ii} x_i^{(k)} = y_i^{(k)}$
- Evaluate $z_i^{(k)} = A_{\Gamma i} x_i^{(k)}$

} <u>In parallel</u>

<u>Property</u>

$$\text{If } \operatorname{cond}(A) = O(h^{-2}) \Rightarrow \operatorname{cond}(S) = O(h^{-1})$$

Iterative methods: iteration-by-subdomains

The original system can be written as:

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} U_1 \\ U_\Gamma \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - A_{\Gamma 2} U_2 - A_{\Gamma\Gamma}^{(2)} U_\Gamma \end{bmatrix}$$

$$A_{22} \quad U_2 \quad = \quad F_2 - A_{2\Gamma} U_\Gamma$$

Consider the iterative scheme:

$$\boxed{\begin{aligned} &\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} U_1^{(k)} \\ U_\Gamma^{(k)} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - A_{\Gamma 2} U_2^{(k-1)} - A_{\Gamma\Gamma}^{(2)} U_\Gamma^{(k-1)} \end{bmatrix} \\ &A_{22} \quad U_2^{(k)} \quad = F_2 - A_{2\Gamma} U_\Gamma^{(\ell)} \end{aligned}}$$

← Neumann conditions

← Dirichlet conditions

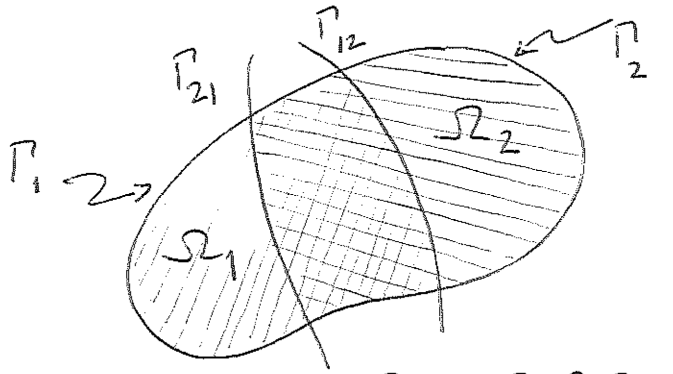
$\ell = k-1$. Jacobi scheme (parallel)
$\ell = k$. Gauss-Seidel scheme (sequential)

- U_1, U_Γ is solved with $A_{\Gamma 2} U_2$ known, i.e., fluxes from Ω_2
- U_2 " " " U_Γ " , i.e., unknown from Ω_1

The scheme considered is the algebraic version of the Dirichlet-Neumann scheme:

$$\begin{aligned} &\mathcal{L} u_1^{(k)} = f \text{ in } \Omega_1 \\ &u_1^{(k)} = \bar{u}|_{\Gamma_1} \text{ on } \Gamma_1 \\ &k \frac{\partial u_1^{(k)}}{\partial n} = k_2 \frac{\partial u_2^{(k-1)}}{\partial n} \text{ on } \Gamma_{12} \end{aligned} \qquad \begin{aligned} &\mathcal{L} u_2^{(k)} = f \\ &u_2^{(k)} = \bar{u}|_{\Gamma_2} \text{ on } \Gamma_2 \\ &u_2^{(k)} = u_1^{(\ell)} \text{ on } \Gamma_{12} \end{aligned}$$

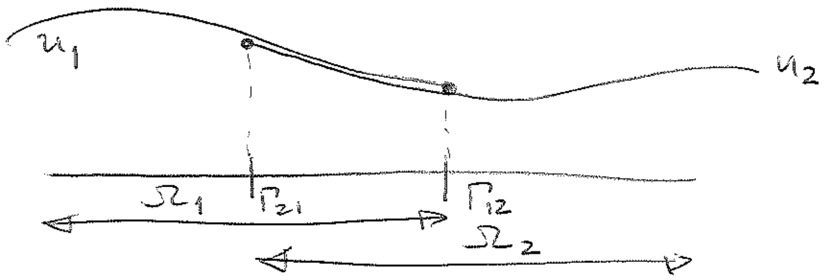
4. SCHWARZ'S METHOD



$$\Omega_{12} = \Omega_1 \cap \Omega_2 = \phi$$

The original problem (with $\bar{u} = 0$ for simplicity) is equivalent to:

$$\left.\begin{aligned} \mathcal{L}u_1 &= f \quad \text{in } \Omega_1 \\ u_1 &= 0 \quad \text{on } \Gamma_1 \\ u_1 &= u_2 \quad \text{on } \Gamma_{12} \end{aligned}\right\} \qquad \left.\begin{aligned} \mathcal{L}u_2 &= f \quad \text{in } \Omega_2 \\ u_2 &= 0 \quad \text{on } \Gamma_2 \\ u_2 &= u_1 \quad \text{on } \Gamma_{21} \end{aligned}\right\}$$



Iteration-by-subdomains:

$$\left.\begin{aligned} \mathcal{L}u_1^{(k)} &= f \quad \text{in } \Omega_1 \\ u_1^{(k)} &= 0 \quad \text{on } \Gamma_1 \\ u_1^{(k)} &= u_2^{(k-1)} \quad \text{on } \Gamma_1 \end{aligned}\right\} \qquad \left.\begin{aligned} \mathcal{L}u_2^{(k)} &= f \quad \text{in } \Omega_2 \\ u_2^{(k)} &= 0 \quad \text{on } \Gamma_2 \\ u_2^{(k)} &= u_1^{(\ell)} \quad \text{on } \Gamma_{21} \end{aligned}\right\}$$

$\ell = k-1$. Jacobi-like scheme: ADDITIVE Schwarz (parallel)

$\ell = k$. Gauss-Seidel-like scheme: MULTIPLICATIVE Schwarz (seq.)

The scheme converges, the speed depending on the width of Ω_{12}

5. DOMAIN DECOMPOSITION METHODS AS PRECONDITIONERS

Recall the equation for the interface degrees of freedom:

$$S U_\Gamma = G \qquad (S: \text{Schur complement})$$

A Richardson iterative scheme to solve this problem with preconditioner P would be:

$$U_\Gamma^{(k)} = U_\Gamma^{(k-1)} + P^{-1}\left(G - S U_\Gamma^{(k-1)}\right), \quad k = 1, 2, \dots$$

If $P = S$, $U_\Gamma^{(1)}$ would be exact, regardless of the initial guess U_Γ^0.

It turns out that

The iteration-by-subdomain schemes correspond to Richardson iterations for the Schur complement system with an appropriate preconditioner P

Let us prove this for the Dirichlet-Neumann scheme. Define:

$$S_1 = A_{\Gamma\Gamma}^{(1)} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma}\,, \quad S_2 = A_{\Gamma\Gamma}^{(2)} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}$$

$$S = S_1 + S_2$$

Consider the Gauss-Seidel-type iteration-by-subdomain:

$$A_{11} U_1^{(k)} = F_1 - A_{1\Gamma} U_\Gamma^{(k)} \Rightarrow U_1^{(k)} = A_{11}^{-1}(F_1 - A_{1\Gamma} U_\Gamma^{(k)})$$

$$A_{22} U_2^{(k-1)} = F_2 - A_{2\Gamma} U_\Gamma^{(k-1)} \Rightarrow U_2^{(k-1)} = A_{22}^{-1}(F_2 - A_{2\Gamma} U_\Gamma^{(k-1)})$$

$$A_{\Gamma 1} U_1^{(k)} + A_{\Gamma\Gamma}^{(1)} U_\Gamma^{(k)} = F_\Gamma - A_{\Gamma 2} U_2^{(k-1)} - A_{\Gamma\Gamma}^{(2)} U_\Gamma^{(k-1)}$$

$$\Rightarrow A_{\Gamma 1} A_{11}^{-1} (F_1 - A_{1\Gamma}) U_\Gamma^{(k)} + A_{\Gamma\Gamma}^{(1)} U_\Gamma^{(k)}$$

$$= F_\Gamma - A_{\Gamma 2} A_{22}^{-1} (F_2 - A_{2\Gamma} U_\Gamma^{(k-1)}) - A_{\Gamma\Gamma}^{(2)} U_\Gamma^{(k-1)}$$

$$\Rightarrow \left(A_{\Gamma\Gamma}^{(1)} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma}\right) U_\Gamma^{(k)}$$

$$= F_\Gamma - A_{\Gamma 1} A_{11}^{-1} F_1 - A_{\Gamma 2} A_{22}^{-1} F_2 - (A_{\Gamma\Gamma}^{(2)} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}) U_\Gamma^{(k-1)}$$

$$\Leftrightarrow S_1 U_\Gamma^{(k)} = G - S_2 U_\Gamma^{(k-1)} = G - S U_\Gamma^{(k-1)} + S_1 U_\Gamma^{(k-1)}$$

$$\Leftrightarrow \boxed{U_\Gamma^{(k)} = U_\Gamma^{(k-1)} + S_1^{-1}(G - S U_\Gamma^{(k-1)})}$$

which is a Richardson iteration for the Schur complement equation with preconditioner $P = S_1$.

It can be shown that the preconditioners for the NN and the RR iteration-by-subdomain schemes are:

$$\text{NN}: \quad P = (\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})^{-1}$$

$$\text{RR}: \quad P = (\gamma_1 + \gamma_2)^{-1} (\gamma_1 I + S_1)(\gamma_2 I + S_2)$$

In the parallel implementation of multi-domain DD schemes, a major problem is to devise preconditioners such that the condition number of $P^{-1}S$ remains bounded when the number of subdomains grows (<u>scalability</u> problem).

6. APPLICATION TO THE NAVIER-STOKES EQUATIONS

Consider the NS equations written as

$$\left.\begin{aligned} \mathcal{N}(u,p) &= \mathbb{f} \quad \text{in } \Omega,\ t>0 \\ u &= 0 \quad \text{on } \partial\Omega,\ t>0 \\ u &= u^0 \quad \text{in } \Omega,\ t=0 \end{aligned}\right\}$$

where

$$\mathcal{N}(u,p) = \begin{bmatrix} \partial_t u + u\cdot\nabla u - \nu\Delta u + \nabla p \\ \nabla\cdot u \end{bmatrix}, \quad \mathbb{f} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

The (pseudo)-stress is now given by

$$t = -pn + \nu\frac{\partial u}{\partial n}$$

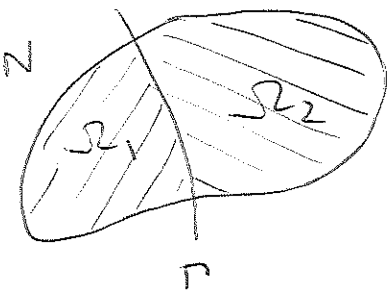
Therefore, an iteration-by-subdomain DN scheme would be (using a Gauss-Seidel-type iteration):

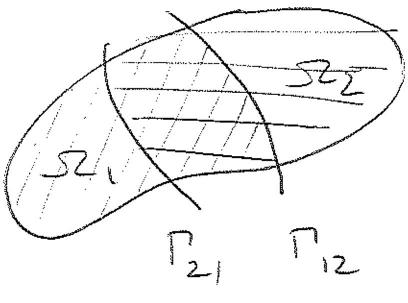
$$\begin{aligned} \mathcal{N}(u_1^{(k)}, p_1^{(k)}) &= \mathbb{f} \quad \text{in } \Omega_1,\ t \text{ fixed},\ t>0 \\ u_1^{(k)}\big|_\Gamma &= u_2^{(k-1)}\big|_\Gamma \quad \text{on } \Gamma,\ t \text{ fixed} \\ \mathcal{N}(u_2^{(k)}, p_2^{(k)}) &= \mathbb{f} \quad \text{in } \Omega_2,\ t \text{ fixed} \\ -p_2^{(k)}n + \nu_2\frac{\partial u_2^{(k)}}{\partial n} &= -p_1^{(k)}n + \nu_1\frac{\partial u_1^{(k)}}{\partial n} \quad \text{on } \Gamma,\ t \text{ fixed}. \end{aligned}$$

whereas a Schwarz's method would be

$$\begin{aligned} \mathcal{N}(u_1^{(k)}, p_1^{(k)}) &= \mathbb{f} \quad \text{in } \Omega_1,\ t \text{ fixed} \\ u_1^{(k)}\big|_\Gamma &= u_2^{(k-1)}\big|_{\Gamma_{12}} \quad \text{on } \Gamma_{12},\ t \text{ fixed} \\ \mathcal{N}(u_2^{(k)}, p_2^{(k)}) &= \mathbb{f} \quad \text{in } \Omega_1,\ t \text{ fixed} \\ u_2^{(k)}\big|_{\Gamma_{21}} &= u_1^{(k)}\big|_{\Gamma_{21}} \quad \text{on } \Gamma_{21},\ t \text{ fixed} \end{aligned}$$

DN





Schwarz