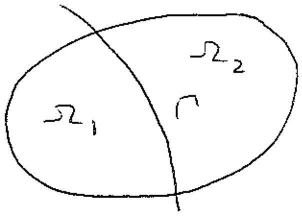
3. COUPLING IN SPACE OF HETEROGENEOUS PROBLEMS

1. MOTIVATION

Suppose that we have two problems in Ω_1 and Ω_2:

$$\mathcal{L}_1 u_1 = f_1 \text{ in } \Omega_1$$
$$\mathcal{L}_2 u_2 = f_2 \text{ in } \Omega_2$$

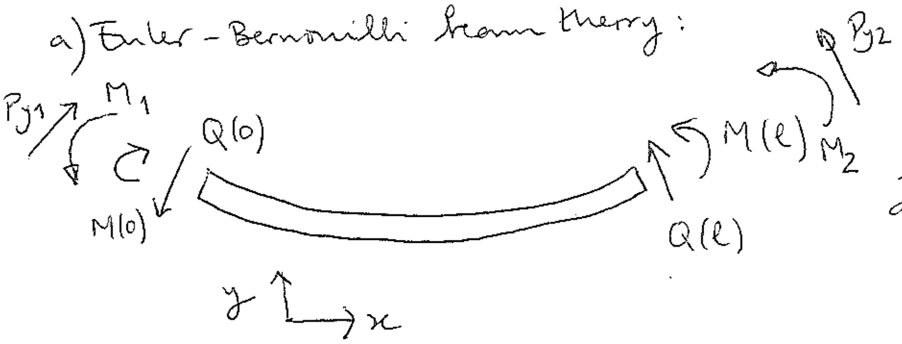


If problems in Ω_1 and Ω_2 are different, there is no single equation $\mathcal{L}u = f$ in Ω. Therefore, the transmission conditions are problem dependent. Usually, they have to be accepted as a physical model. This is so both at the continuous and at the discrete level.

In what follows, we consider examples of heterogeneous problems. There are many in the applications.

2 COUPLING OF THICK AN THIN BEAMS

a) Euler-Bernouilli beam theory:



Field equation:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2 v}{dx^2}\right) = q$$

Dirichlet boundary conditions:

$$v\Big|_\Gamma = \bar{v} \quad , \quad \frac{dv}{dx}\Big|_\Gamma = \bar{\theta}$$

Neumann boundary conditions:

$$Q(x)\Big|_\Gamma = -\frac{d}{dx}\left(EI\frac{d^2 v}{dx^2}\right)\Big|_\Gamma = \bar{Q} \quad , \quad M(x)\Big|_\Gamma = EI\frac{d^2 v}{dx^2}\Big|_\Gamma = \bar{M}$$

b) Timoshenko's beam theory

Field equations:

$$\left.\begin{aligned} \frac{d}{dx}\left(EI\frac{d\theta}{dx}\right) + GA^*\left(\frac{dv}{dx}-\theta\right) &= 0 \\ \frac{d}{dx}\left[GA^*\left(\frac{dv}{dx}-\theta\right)\right] &= -q \end{aligned}\right\}$$

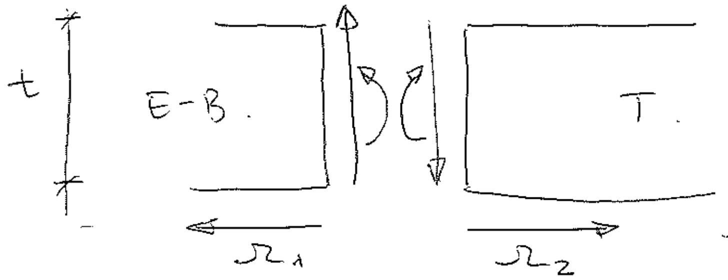
Dirichlet boundary conditions:

$$v\big|_\Gamma = \bar{v}\,, \qquad \theta\big|_\Gamma = \bar{\theta}$$

Neumann BCs:

$$Q(x)\big|_\Gamma = GA^*\left(\frac{dv}{dx}-\theta\right)\Big|_\Gamma = \bar{Q}\,, \quad M(x)\big|_\Gamma = EI\frac{d\theta}{dx}\Big|_\Gamma = \bar{M}$$

c) Transmission conditions



$$v_1 = v_2$$
$$\frac{dv_1}{dx} = \theta_2$$
$$-\frac{d}{dx}\left(EI\frac{d^2v_1}{dx^2}\right) = GA^*\left(\frac{dv_2}{dx}-\theta_2\right)$$
$$EI\frac{d^2v_1}{dx^2} = EI\frac{d\theta_2}{dx}$$

Dirichlet-to-Neumann operators (DtN):

$$\bar{v},\bar{\theta} \text{ on } \Gamma \xrightarrow{\text{Solve}} v_1 \text{ in } \Omega_1 \xrightarrow{\text{Restrict}} Q_1, M_1 \text{ on } \Gamma$$
$$\xrightarrow{\text{Solve}} v_2, \theta_2 \text{ in } \Omega_2 \xrightarrow{\text{Restrict}} Q_2, M_2 \text{ on } \Gamma$$

If $S_i(\bar{v},\bar{\theta}) = (Q_i, M_i)$, $i=1,2$, the Steklov-Poincaré problem is: find $(\bar{v},\bar{\theta})$ defined on Γ such that

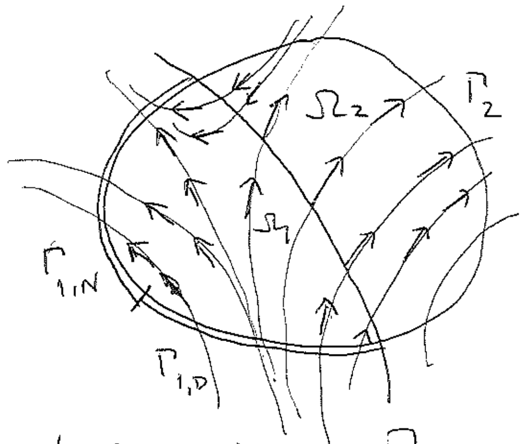
$$S_1(\bar{v},\bar{\theta}) = S_2(\bar{v},\bar{\theta}).$$

A way to obtain the transmission conditions:

- Write the Timoshenko equations on Ω_1 and Ω_2.
- Write the transmission conditions.
- Take the limit $t \to 0$ in Ω_1 only (ASYMTOTIC ANALYSIS)

3. COUPLING OF CONVECTION AND CONVECTION-DIFFUSION

Problem:



$$\mathcal{L}_1 u_1 = a\cdot\nabla u_1 + s u_1 = f \text{ in } \Omega_1$$

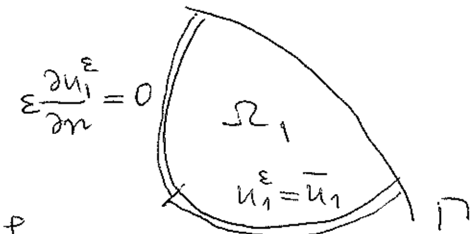
$$\mathcal{L}_2 u_2 = -k\Delta u_2 + a\cdot\nabla u_2 + s u_2 = f \text{ in } \Omega_2$$

$k>0, \quad \nabla\cdot a = 0, \quad s\geq 0$

BCs:

- $u_1 = \bar{u}_1$ on $\Gamma_{1,D} = \{x\in\Gamma_1 \mid n\cdot a \leq 0\}$ (inflow)
- No condition on $\Gamma_{1,N} = \{x\in\Gamma_1 \mid n\cdot a > 0\}$ (outflow).
- $u_2 = \bar{u}_2$ on $\Gamma_{2,D}$
- $k\frac{\partial u_2}{\partial n} = \bar{q}_2$ on $\Gamma_{2,N}$

Interface conditions



Consider

$$\mathcal{L}_1^\varepsilon u_1^\varepsilon = -\varepsilon\Delta u_1^\varepsilon + a\cdot\nabla u_1^\varepsilon + s u_1^\varepsilon = f$$

since $\nabla\cdot a = 0$, $\quad -\varepsilon\Delta u + a\cdot\nabla u = \nabla\cdot(-\varepsilon\nabla u + a u)$

The weak form of the problem is:

$$\int_{\Omega_1} \delta u_1\left(-\varepsilon\Delta u_1^\varepsilon + a\cdot\nabla u_1^\varepsilon + s u_1^\varepsilon\right)$$

$$= \varepsilon\int_{\Omega_1}\nabla\delta u_1\cdot\nabla u_1^\varepsilon - \int_{\Omega_1}\nabla\delta u_1\cdot a u_1 + \int_{\partial\Omega_1}\delta u_1\left(-\varepsilon\frac{\partial u_1^\varepsilon}{\partial n} + a\cdot n u_1^\varepsilon\right)$$

$$+\int_{\Omega_1}\delta u_1 s u_1^\varepsilon = \int_{\Omega_1}\delta u_1 f$$

and similarly on Ω_2. The interface conditions are:

$$u_1^\varepsilon = u_2, \quad \varepsilon\frac{\partial u_1^\varepsilon}{\partial n} - a\cdot n u_1^\varepsilon = k\frac{\partial u_2}{\partial n} - a\cdot n u_2 \text{ on } \Gamma.$$

It can be shown that $u_1^\varepsilon \xrightarrow{\varepsilon\to 0} u_1$ in $L^2(\Omega_1)$. Therefore:

$$\boxed{\begin{aligned} u_1 &= u_2 \text{ on } \Gamma_{in} = \{x\in\Gamma \mid a\cdot n_1 < 0\} \\ -a\cdot n\, u_1 &= k\frac{\partial u_2}{\partial n} - a\cdot n u_2 \text{ on } \Gamma \end{aligned}}$$

Problem in Ω_1 is hyperbolic, whereas the problem in Ω_2 is elliptic.

Dirichlet – Neumann (Robin) IBS

Problem in Ω_1

$$\mathcal{L}_1 u_1^{(k)} = f \quad \text{in } \Omega_1$$
$$u_1^{(k)} = \bar{u}_1 \quad \text{on } \Gamma_{1,D}$$
$$\text{—}$$
$$u_1^{(k)} = u_2^{(k-1)} \quad \text{on } \Gamma_{in}$$
$$\text{—}$$

Problem in Ω_2

$$\mathcal{L}_2 u_2^{(k)} = f \quad \text{in } \Omega_2$$
$$u_2^{(k)} = \bar{u}_2 \quad \text{on } \Gamma_{2,D}$$
$$k \frac{\partial u_2^{(k)}}{\partial n} = \bar{q}_2 \quad \text{on } \Gamma_{2,N}$$
$$k \frac{\partial u_2^{(k)}}{\partial n} = 0 \quad \text{on } \Gamma_{in}$$
$$k \frac{\partial u_2^{(k)}}{\partial n} - a\cdot n\, u_2^{(k)} = -a\cdot n\, u_1^{(k)} \quad \text{on } \Gamma \setminus \Gamma_{in}$$

(Gauss – Seidel option)

Remark: $u_1^{(k)} = u_2^{(k-1)}$ on Γ_{in} may be replaced by

$$u_1^{(k)} = \theta u_2^{(k-1)} + (1-\theta)\, u_1^{(k-1)}$$

on θ optimized (underrelaxation).

4. COUPLING OF THE STOKES AND THE DARCY PROBLEMS

Stokes' problem:

$$\left.\begin{aligned} -\nu \Delta u_s + \nabla p_s &= f \\ \nabla \cdot u_s &= 0 \end{aligned}\right\}$$

Darcy's problem

$$\left.\begin{aligned} u_D + K \nabla \varphi &= 0 \\ \nabla \cdot u_D &= 0 \end{aligned}\right\}$$

φ: potential.

The weak form of these problems uses the fact that:

$$\int_{\Omega_s} \delta u_s \cdot (-\nu \Delta u_s + \nabla p_s) = \int_{\Omega_s} \nu \nabla \delta u_s : \nabla u_s - \int_{\Omega_s} p_s \nabla \cdot \delta u_s - \int_{\partial\Omega_s} \delta u_s \cdot \left[n \cdot \left(-p_s I + \nu \nabla^S u_s \right) \right]$$

$$\int_{\Omega_D} \delta u_D \cdot (K^{-1} u_D + \nabla \varphi) = \int_{\Omega_D} \delta u_D \cdot K^{-1} u_D - \int_{\Omega_D} \varphi \nabla \cdot \delta u_D + \int_{\partial\Omega_D} \varphi\, n \cdot \delta u_D$$

The Stokes problem is well defined (in weak form) if:

$$u_S \in H^1(\Omega_S)^d, \quad p_S \in L^2(\Omega_S) \quad (d: \text{space dimension})$$

The Darcy problem is well posed if (DUAL FORM of Darcy's problem):

$$u_D \in H(\mathrm{div}, \Omega_D) = \{ v : \Omega_D \to \mathbb{R}^d \mid v \in L^2(\Omega_D)^d, \nabla\cdot v \in L^2(\Omega_D) \}$$

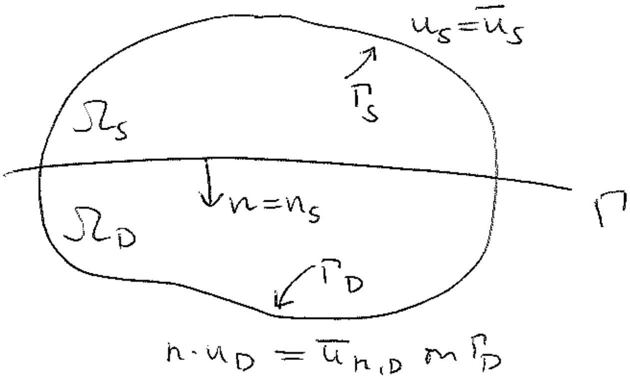
$$\varphi \in L^2(\Omega_D)$$

$$\boxed{\text{Fact: if } v \in H(\mathrm{div}, \Omega) \Rightarrow [\![n\cdot v]\!]_\Gamma = 0 \text{ a.e.}}$$

(Recall: if $v \in H^1(\Omega)^d \Rightarrow [\![v]\!]_\Gamma = 0$ a.e.)

We have:

$$\left.\begin{array}{l} u_D \in H(\mathrm{div}, \Omega_D) \\ u_S \in H^1(\Omega_S) \subset H(\mathrm{div}, \Omega_S) \end{array}\right\} \Rightarrow \boxed{n\cdot u_D = n\cdot u_S} \text{ on } \Gamma$$



Let us take δu_S, δu_D such that

$$n\cdot \delta u_S = n \cdot \delta u_D$$

If, in 2D:

$$\delta u_S = \delta u_S^n \, n + \delta u_S^t \, t$$

$$\delta u_D = \delta u_D^n \, n + \delta u_D^t \, t$$

then $\delta u_S^n = \delta u_D^n$.

We have that:

$$\delta u_S \cdot [n\cdot(p_S I - \nu \nabla u_S)] = \underline{p_S \delta u_S^n - \nu\, n\cdot\nabla u_S \cdot n\, \delta u_S^n} - \nu\, n\cdot\nabla u_S \cdot t\, \delta u_S^t$$

$$\varphi\, n\cdot \delta u_D = \underline{\varphi\, \delta u_D^n}$$

From where:

$$\boxed{\varphi = p_S - \nu\, n\cdot \nabla u_S \cdot n \quad \text{on } \Gamma} \quad \text{(Other scalings are possible)}$$

It remains to determine the condition for the tangential component of the velocity field, which needs <u>not</u> being continuous. The most common choice is:

$$\boxed{\nu t \cdot (n\cdot \nabla u_S) = -\frac{\alpha_{BJ}}{\sqrt{K}} (u_S - u_D)\cdot t} \quad \text{on } \Gamma$$

Beavers - Joseph transmission condition

It can be justified:

- Experimentally (Saffman)
- Through asymptotic analysis.

Remark. It turns out that $|u_D| \ll |u_S|$, and $(u_S - \nu u_D)\cdot t \approx u_S \cdot t$.

Problem in Ω_S

$$-\nu \Delta u_S + \nabla p_S = f$$
$$\nabla \cdot u_S = 0$$
$$u_S = \overline{u}_S \text{ on } \Gamma_S$$

Problem in Ω_D

$$\left.\begin{aligned} K^{-1} u_D + \nabla \varphi &= 0 \\ \nabla \cdot u_D &= 0 \end{aligned}\right\}$$
$$n \cdot u_D = \overline{u}_{n,D} \text{ on } \Gamma_D$$

"STRONGLY" $\longrightarrow$ $n \cdot u_S = n \cdot u_D$

"WEAKLY" $\longrightarrow$ $p_S - n \cdot \nu \nabla u_S \cdot n = \varphi$ $\Big\}$ on Γ

$\hookrightarrow$ $t \cdot u_S = -\dfrac{\sqrt{K}}{\alpha_{BJ}}\, t\, \nu (n \cdot \nabla u_S)$ on Γ (Robin-type for $t \cdot u_S$)

Steklov - Poincaré problem

Dirichlet-to-Neumann operators:

$$\lambda \;(= n \cdot u_S|_\Gamma) \xrightarrow{\text{Solve}} [u_S, p_S] \xrightarrow{\text{Evaluate and restrict}} \mathcal{S}_S(\lambda) = p_S - n \cdot \nu \nabla u_S n|_\Gamma$$

$$\lambda \;(= n \cdot u_D|_\Gamma) \longrightarrow [u_D, \varphi] \longrightarrow \mathcal{S}_D(\lambda) = \varphi|_\Gamma$$

SP problem: $\mathcal{S}_S(\lambda) = \mathcal{S}_D(\lambda)$

Dirichlet - Neumann IBS

$$\left.\begin{aligned} -\nu \Delta u_S^{(k)} + \nabla p_S^{(k)} &= f \\ \nabla \cdot u_S^{(k)} &= 0 \end{aligned}\right\} \text{ in } \Omega_S$$
$$u_S^{(k)} = \overline{u}_S \text{ on } \Gamma_S$$
$$n \cdot u_S^{(k)} = n \cdot u_D^{(k-1)} \text{ on } \Gamma$$
$$t \cdot u_S^{(k)} = -\frac{\sqrt{K}}{\alpha_{BJ}}\, t \cdot \nu\, n \cdot \nabla u_S^{(k)} \text{ on } \Gamma$$

$$\left.\begin{aligned} K^{-1} u_D^{(k)} + \nabla \varphi &= 0 \\ \nabla \cdot u_D^{(k)} &= 0 \end{aligned}\right\} \text{ in } \Omega_D$$
$$n \cdot u_D^{(k)} = \overline{u}_{n,D} \text{ on } \Gamma_D$$
$$\varphi^{(k)} \underset{\text{weakly}}{=} p_S - n \cdot \nu \nabla u_S^{(k)} n \text{ on } \Gamma$$

Gauss - Seidel-type IBS