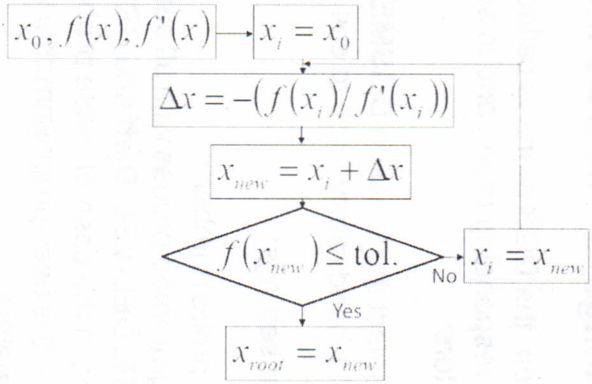
Numerical methods for PDE — Exercises: Basics

Written by: Mauricio Barrera **Date:** November 2016

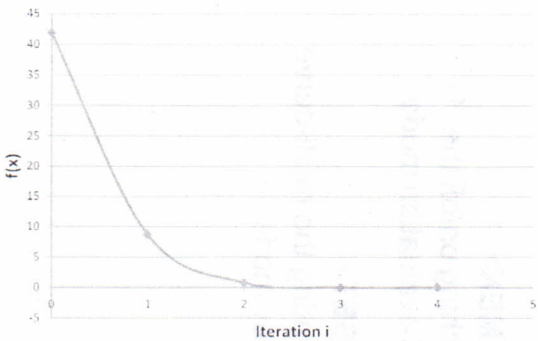
Problem 1. Given the function $f(x) := x^3 + 2x^2 + 10x - 20$, four iterations of the Newton method will be applied to find the root of $f(x)$, using $\sqrt[3]{20} \approx 2.714$ as starting guess.

At the root value $x_{root}, f(x) = 0$. The linear approximation to $f(x_{root})$ from any value x_i reads as $f(x_{root}) = 0 = f(x_i) + f'^{(x_i)}\Delta x \rightarrow \Delta x = -\big(f(x_i)/f'(x_i)\big)$.
Therefore, the generation of a value x_{new} closer to x_{root} than x_i is comes from $x_{new} = x_i + \Delta x$.
The corresponding flow diagram is set out to the right:



A chart showing the implementation results and a plot of the convergence –that is, $f(x_i)$ versus iteration i is set out next. The approximated value for x_{root} can be taken a 1.37.

i	x_i	$f(x_i)$	$f'(x_i)$	Δx
0	2,714418	41,8803	42,96186	-0,97483
1	1,739592	8,712611	26,03692	-0,33463
2	1,404967	0,770847	21,54167	-0,03578
3	1,369183	0,007912	21,10072	-0,00037
4	1,368808	8,59E-07		



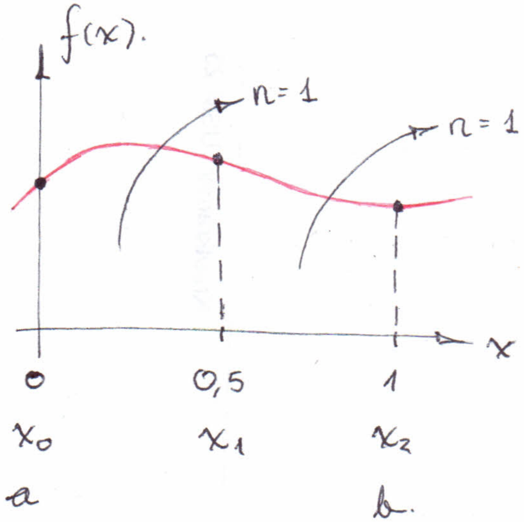
Problem 5. The required order q of the quadrature is 3, so the number of integration points, in principle, depends on the specific scheme one intends applying:

- Newton-Cotes formulae with an even number of terms n. Here $q = n + 1$; but this is just the same number as the degrees of freedom in this method, which corresponds to the number of integration points used $\{x_i\}_0^n$. Hence one needs three integration points.
- Newton-Cotes formulae with an odd number of terms n. Here $q = n$; however, as the number of degrees of freedom is equal to $n + 1$ then one needs four integration points.

- Gauss quadrature method. Here the number of degrees of freedom (made up by *both* integration points and weights) is given by $2n + 2$, whereas the quadrature order is $q = 2n + 1$. Therefore, if $q = 3$ then we have that $4 = 2n + 2$. But the number of integration points is half the degree of freedom, or $n + 1$. Thus, dividing by two the previous equation we have $2 = n + 1$ and conclude that the method requires two integration points.

Problem 6. a) In the Gaussian quadrature method the polynomial exactly integrated is of order $2n + 1$. Hence, given $n + 1$ integration points, said order is given by $2(n + 1) - 1$. In other words, the polynomial exactly integrated is of order equal to twice the number of integration points minus one. b) For $n = 2$ then the order of the polynomial exactly integrated if of order up to and including $2n + 1 = 5$. Therefore integrals ii and iii can be integrated exactly.

Problem 7. i) Trapezoidal rule over two uniform intervals



$\Big\}\, m = 2.$

for each subinterval:

$$h = \frac{b-a}{m \cdot n} = \frac{1-0}{2} = 0,5$$

- For $I = \int_0^1 12x\,dx$, $f(x) = 12x$ and the composite trapezoidal rule gives: $I \simeq \frac{h}{2}\left(f(x_0) + 2f(x_1) + f(x_2)\right) = 6.$

Exact result: 6; error $= -[(b-a)^3/12m^2]\,f''(\mu) = -[1/12 \cdot 2^2] \cdot 0 = 0.$

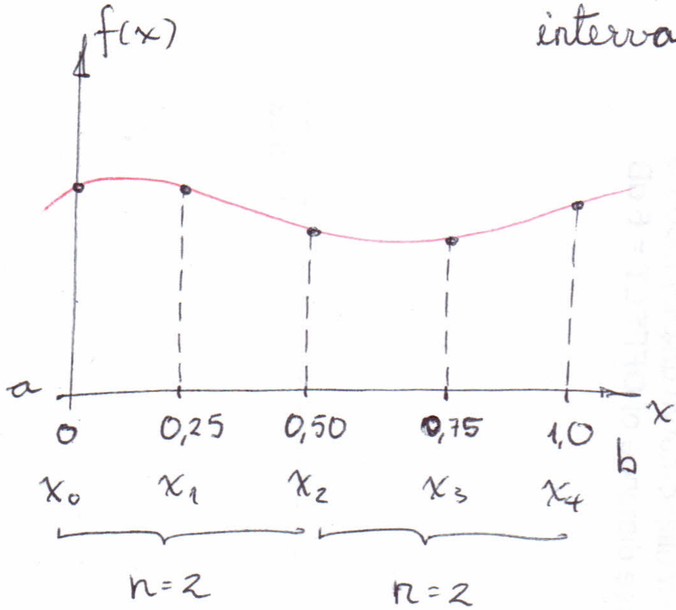
- For $I = \int_0^1 (5x^3 + 2x)\,dx$, $f(x) = 5x^3 + 2x$ and:

$$I \simeq \frac{h}{2}\left(f(x_0) + 2f(x_1) + f(x_2)\right) = 2,5625.$$

Exact result: 2.25; error $= -[1/12 \cdot 2^2] \cdot 30\mu$; at $\mu = 1$, error $= -0,625.$

This is twice as large as the actual error $(2,25 - 2,5625 = -0,3125)$.

ii) Simpson's rule over two uniform intervals



$m=2$. for each subinterval:

$$h=\frac{b-a}{m\cdot n}=\frac{1-0}{2\cdot 2}=0{,}25.$$

- For $I=\int_0^1 (5x^3+2x)\,dx$, $f(x)=5x^3+2x$ and:

$$I\simeq \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]+\frac{h}{3}\left[f(x_2)+4f(x_3)+f(x_4)\right]=2{,}25.$$

Exact result: 2,25 ; error $=-\frac{mh^5}{90}f''''(\mu)$; $f''''(x)=0\rightarrow$ error $=0$.

This makes sense, since for Simpson's rule, $n=2$ (even), and $q=n+1=3$, which is the order of the polynomial that can be integrated exactly.

- For $I=\int_0^1 12x\,dx$ we have:

$$I\simeq \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]+\frac{h}{3}\left[f(x_2)+4f(x_3)+f(x_4)\right]=6.$$

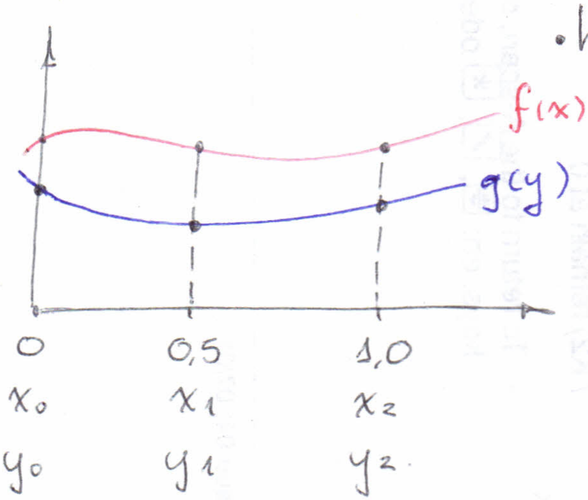
Exact result: 6 ; error $=-\frac{mh^5}{90}f''''(\mu)$; $f''''(x)=0\rightarrow$ error $=0$.

Problem #10. $\int_0^1 \int_0^1 (9x^3+8x^2)(y^3+y)\,dx\,dy = I$.

Integrating ~~over~~ x first: $\int_0^1 (y^3+y)\left[\int_0^1 (9x^3+8x^2)\,dx\right]dy = I$.

$$I = \int_0^1 (y^3+y)\left[\frac{59}{12}\right]dy = \left[\frac{59}{12}\right]\int_0^1 (y^3+y)\,dy = \left[\frac{59}{12}\right]\cdot\left[\frac{3}{4}\right] \approx 3{,}688$$

Simpson's rule:



• $h = 0{,}5$ • $f(x) = 9x^3+8x^2$ • $g(y) = y^3+y$

Integration over x: $I_x \simeq \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right] = 4{,}916 \approx \frac{59}{12}$

Integration over y: $I_y \simeq \frac{h}{3}\left[g(y_0)+4g(y_1)+g(y_2)\right] = 0{,}75 = \frac{3}{4}$.

Then: $I \simeq I_x \cdot I_y = 4{,}916 \cdot 0{,}75 = 3{,}687$

Error: $\left[-\frac{h^5}{90} f''''(\mu)\right]\cdot\left[-\frac{h^5}{90} g''''(\mu)\right] = 0$, for $f''''(x) = 0$

$g''''(y) = 0$.

Hence, the approximation behaves as expected.