Homework

Arnab Samaddar Chaudhuri

2) $Tds = de + pdV$ for all infinitesimal changes

$V = \frac{1}{\rho}$ [here V is specific gravity]

$dV = -\frac{1}{\rho^2} d\rho$

$\therefore Tds = de - \frac{p}{\rho^2} d\rho$

Taking material derivative

$$T\frac{Ds}{Dt} = \frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} \Rightarrow \rho T \frac{Ds}{Dt} = \rho\frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}$$

$$\Rightarrow \rho T\frac{Ds}{Dt} = \sigma : \nabla v - \nabla \cdot q - \frac{p}{\rho}\frac{D\rho}{Dt}$$

$$= -p\nabla\cdot v - \nabla\cdot q + \phi - \frac{p}{\rho}\frac{D\rho}{Dt}$$

where $\phi = \lambda(\nabla\cdot v)^2 + 2\mu \nabla^s v : \nabla v$

ϕ is always ≥ 0 since $\mu > 0$, $K \geq 0$

Here we can show that $\nabla^s v : \nabla v$ is positive

$$\nabla^s v : \nabla v = \nabla^s v : \frac{1}{2}(\nabla^s v + \nabla^w v)$$

$$= \frac{1}{2}\left[(\nabla^s v)^2 + \cancel{\nabla^s v : \nabla^w v}^{\,0}\right]$$

Now,

$$\rho T\frac{Ds}{Dt} = -p\nabla\cdot v - \nabla\cdot q + \phi - \frac{p}{\rho}\frac{D\rho}{Dt}$$

$$= -\frac{p}{\rho}\left(\cancel{\frac{D\rho}{Dt} + \rho\nabla\cdot v}^{\,0}\right) - \nabla\cdot q + \phi$$

$$\Rightarrow \rho T\frac{Ds}{Dt} = -\nabla\cdot q + \phi$$

$$\Rightarrow \rho\frac{Ds}{Dt} = -\frac{\nabla\cdot q}{T} + \frac{\phi}{T}$$

$$= -\nabla\cdot\left(\frac{q}{T}\right) - q\cdot\frac{\nabla T}{T^2} + \frac{\phi}{T}$$

Now,

$$\nabla\cdot\left(\frac{q}{T}\right) = \frac{1}{T}\nabla q - q\cdot\frac{\nabla T}{T^2}$$

$$= -\nabla\cdot\left(\frac{q}{T}\right) + \frac{K\,\nabla T\cdot\nabla T}{T^2} + \frac{\phi}{T}$$

$$= \nabla\cdot\left(\frac{q}{T}\right) + \underbrace{\frac{\phi}{T} + \frac{k(\nabla T)^2}{T^2}}_{A}$$

where A is always ≥ 0 ~~since~~ since $\frac{\phi}{T}$ ~~≥ 0 and~~ $k > 0$

The term A can be zero if ϕ and gradient of temperature both are zero

$$\rho\frac{Ds}{Dt} = -\nabla\cdot\left(\frac{q}{T}\right) + A$$

Taking the global form

$$\int_{V_t} \rho\frac{Ds}{Dt}\,dV = -\int_{V_t}\nabla\cdot\left(\frac{q}{T}\right)dV + \int_{V_t} A\,dV$$

$$\Rightarrow \frac{D}{Dt}\int_{V_t}\rho s\,dV = -\int_{V_t}\nabla\cdot\left(\frac{q}{T}\right)dV + \int_{V_t} A\,dV \quad \text{[Using the concept of Reynolds Lemma]}$$

$$\Rightarrow \frac{D}{Dt}\int_{V_t}\rho s\,dV = -\int_{S_t}\frac{q\cdot n}{T}\,dS + \int_{V_t} A\,dV \quad \text{[Using divergence theorem]}$$

$\Downarrow$ This term is always ≥ 0

$$\therefore \frac{D}{Dt}\int_{V_t}\rho s\,dV \geq -\int_{S_t}\frac{q\cdot n}{T}\,dS$$

1a) $$\nabla\cdot(\nabla\times F) = (\varepsilon_{ijk}F_{k,j})_{,i} = \varepsilon_{ijk}F_{k,ji} = \varepsilon_{123}F_{3,21} + \varepsilon_{312}F_{2,13} + \varepsilon_{231}F_{1,32} + \varepsilon_{213}F_{3,12} + \varepsilon_{321}F_{1,23} + \varepsilon_{132}F_{2,31}$$

$$= 0$$

b) $$\nabla\times(\nabla\times F) = \nabla\times(\varepsilon_{klm}F_{m,l}) = \varepsilon_{ijk}\varepsilon_{klm}F_{m,lj} = \varepsilon_{ijk}\varepsilon_{lmk}F_{m,lj}$$

~~$(\delta_{il} - \delta_{jm}$~~

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})F_{m,lj}$$

$$= F_{j,ij} - F_{i,jj}$$

$$= \nabla(\nabla\cdot F) - \nabla^2 F$$

c) $\nabla \cdot (F \times G) = \nabla \cdot (\varepsilon_{jki} F_j G_k)$

$$= \varepsilon_{jki} F_{j,i} G_k + \varepsilon_{jki} F_j G_{k,i}$$

$$= G_k \varepsilon_{kij} F_{j,i} - F_i \varepsilon_{jik} G_{k,i}$$

$$= G \cdot \nabla \times F - F \cdot \nabla \times G$$