ADVANCED FLUID MECHANICS

Master of Science in Computational Mechanics/ Numerical Methods

Fall Semester 2015

Homework 2: Dimensional analysis, compressible flow and Navier-Stokes equations

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Exercise 1·

1. a) The dimensional matrix for the variables on this case is:

	$\Delta P/L$	ρ	$\bar{v}_0$	R	R_1	μ_1	μ_2	σ
$\mathcal{M}$	1	1	0	0	0	1	1	1
$\mathcal{L}$	-2	-3	1	1	1	-1	-1	0
$\mathcal{T}$	-2	0	-1	0	0	-1	-1	-1

Taking into consideration that the rank of the matrix r is 3, and the number of variables is 8, we know that we can obtain 5 different Π products.

Selecting as primary variables $\bar{v}_0$, R_1 and ρ :

$$\Pi_1 = \rho^a \, (\bar{v}_0)^b \, (R_1)^c \, \frac{\Delta P}{L}$$

$$\mathcal{M}^0\mathcal{L}^0\mathcal{T}^0 = \mathcal{M}^a\mathcal{L}^{-3a}\mathcal{L}^b\mathcal{T}^{-b}\mathcal{L}^c\mathcal{M}^1\mathcal{L}^{-2}\mathcal{T}^{-2}$$

Solving: $a = -1;\ b = -2;\ c = 1 \rightarrow \boxed{\Pi_1 = \frac{\Delta P}{L}\frac{\mathrm{R}_1}{\rho\bar{v}_0^2}}$

$$\Pi_2 = \rho^a \, (\bar{v}_0)^b \, (R_1)^c \, R$$

$$\mathcal{M}^0\mathcal{L}^0\mathcal{T}^0 = \mathcal{M}^a\mathcal{L}^{-3a}\mathcal{L}^b\mathcal{T}^{-b}\mathcal{L}^c\mathcal{L}^1$$

Solving: $a = 0;\ b = 0;\ c = -1 \rightarrow \boxed{\Pi_2 = \frac{R}{R_1}}$

$$\Pi_3 = \rho^a \, (\bar{v}_0)^b \, (R_1)^c \, \mu_1$$

$$\mathcal{M}^0\mathcal{L}^0\mathcal{T}^0 = \mathcal{M}^a\mathcal{L}^{-3a}\mathcal{L}^b\mathcal{T}^{-b}\mathcal{L}^c\mathcal{M}^1\mathcal{L}^{-1}\mathcal{T}^{-1}$$

Solving: $a = -1;\ b = -1;\ c = -1 \rightarrow \Pi_3 = \frac{\mu_1}{\rho\bar{v}_0 R_1} \xrightarrow{inverting} \boxed{\Pi_3 = \frac{\rho\bar{v}_0 R_1}{\mu_1}}$

$$\Pi_4 = \rho^a\,(\bar{v}_0\,)^b\,(R_1)^c\,\mu_2$$

$$\mathcal{M}^0\mathcal{L}^0\mathcal{T}^0 = \mathcal{M}^a\mathcal{L}^{-3a}\mathcal{L}^b\mathcal{T}^{-b}\mathcal{L}^c\mathcal{M}^1\mathcal{L}^{-1}\mathcal{T}^{-1}$$

Solving: $a = -1;\ b = -1;\ c = -1 \rightarrow\ \Pi_4 = \frac{\mu_2}{\rho\bar{v}_0R_1} \xrightarrow{inverting} \boxed{\Pi_4 = \frac{\rho\bar{v}_0R_1}{\mu_2}}$

$$\Pi_5 = \rho^a\,(\bar{v}_0\,)^b\,(R_1)^c\,\sigma$$

$$\mathcal{M}^0\mathcal{L}^0\mathcal{T}^0 = \mathcal{M}^a\mathcal{L}^{-3a}\mathcal{L}^b\mathcal{T}^{-b}\mathcal{L}^c\mathcal{M}^1\mathcal{T}^{-1}$$

Solving: $a = -1;\ b = -2;\ c = -1 \rightarrow\ \Pi_5 = \frac{\sigma}{\rho\bar{v}_0^2R_1} \xrightarrow{inverting} \boxed{\Pi_5 = \frac{\rho\bar{v}_0^2R_1}{\sigma}}$

Finally we have that:

$$\Pi_1 = \mathcal{F}(\Pi_2, \Pi_3, \Pi_4, \Pi_5)$$

$$\frac{\Delta P}{L} = \frac{\rho\bar{v}_0^2}{\mathrm{R}_1}\mathcal{F}(\Pi_2, \Pi_3, \Pi_4, \Pi_5)$$

1. b) From the dimensionless numbers obtained in the previous section we can identify that Π_3 and Π_4 correspond to the Reynolds number associated to each fluid; and that Π_5 is the Weber number.

The Weber number is used to characterize the formation of droplets and bubbles (when We>1). Understanding that in our case the formation of a droplet or bubble may occur when a wave on the interface grows enough to fold over itself isolating part of one fluid inside the other, we may ensure that the flow is far enough from the formation of droplets, hence:

$$\frac{\rho\bar{v}_0^2R_1}{\sigma} \ll 1$$

1. c) Analogous to Weber number, which compares inertial forces versus surface tension, we can find the Eötvös number which compares body forces versus surface tension. This dimensionless number is defined as:

$$Eo = \frac{\Delta\rho g L^2}{\sigma}$$

For the case of study the difference of density between the two fluids is close to 0, and therefore $Eo \approx 0$. The latter results in the fact that body forces (gravity) cannot be source of the wave formation when the densities are similar.

1. d) Taking into account that $V_r = 0, V_\theta = 0\ and\ V_z = f(r)$; the Navier-Stokes equations in cylindrical coordinates remain:

$$0 = -\frac{\partial p}{\partial r} \tag{1}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} \tag{2}$$

$$0 = -\frac{\partial p}{\partial z} + \mu\,\nabla^2 V_z \tag{3}$$

Where is important to remark that the mass conservation equation is automatically verified and that p is only a function z (due to equations (1) and (2)).

Considering $\partial p/\partial z = -\Delta p/L$ and applying the Laplacian in cylindrical coordinates we proceed from (3) to:

$$-\frac{\Delta p}{L} = \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right)\right) \tag{4}$$

As we have two fluids with different viscosities we need to solve the previous differential equation for both regions,

$$-\frac{\Delta p}{L} = \mu_2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_{2z}}{\delta r}\right)\right); \quad -\frac{\Delta p}{L} = \mu_1\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_{1z}}{\partial r}\right)\right)$$

and therefore we need 4 Boundary conditions:

- Non-slip condition on the wall: $V_{2z}(R) = 0$
- Same velocity at the interface: $V_{1z}(R_1) = V_{2z}(R_1)$
- Same shear stress at the interface: $\mu_1 \frac{\partial V_{1z}}{\partial r}\Big|_{r=R_1} = \mu_2 \frac{\partial V_{2z}}{\partial r}\Big|_{r=R_1}$
- Velocity at r=0 must be finite

1. e) Integrating two times the differential equations we obtain:

$$-\frac{\Delta p}{4\mu_2 L}r^2 + C_1\ln(r) + c_2 = V_{2z} \tag{5}$$

$$-\frac{\Delta p}{4\mu_1 L}r^2 + C_3\ln(r) + c_4 = V_{1z} \tag{6}$$

Where the integration constants can be obtained after applying the boundary conditions:

$$c_1 = 0; \quad C_2 = \frac{\Delta p}{4\mu_2 L}R^2; \quad C_3 = 0\,; \quad C_4 = \left(\frac{R^2 - R_1^2}{\mu_2} + \frac{R_1^2}{\mu_1}\right)\frac{\Delta p}{4L}$$

Being the final expression for V_z :

$$\boxed{V_Z = \begin{cases} \dfrac{\Delta p}{4L}\left(\dfrac{R^2 - R_1^2}{\mu_2} + \dfrac{R_1^2 - r^2}{\mu_1}\right) & for\ \ 0 < r < R_1 \\ \dfrac{\Delta p}{4\mu_2 L}(R^2 - r^2) & for\ \ R_1 \le r < R \end{cases}} \tag{7}$$

With velocity at the interface:

$$\boxed{V_Z(R_1) = \frac{\Delta p}{4\mu_2 L}\left(R^2 - {R_1}^2\right)} \tag{8}$$

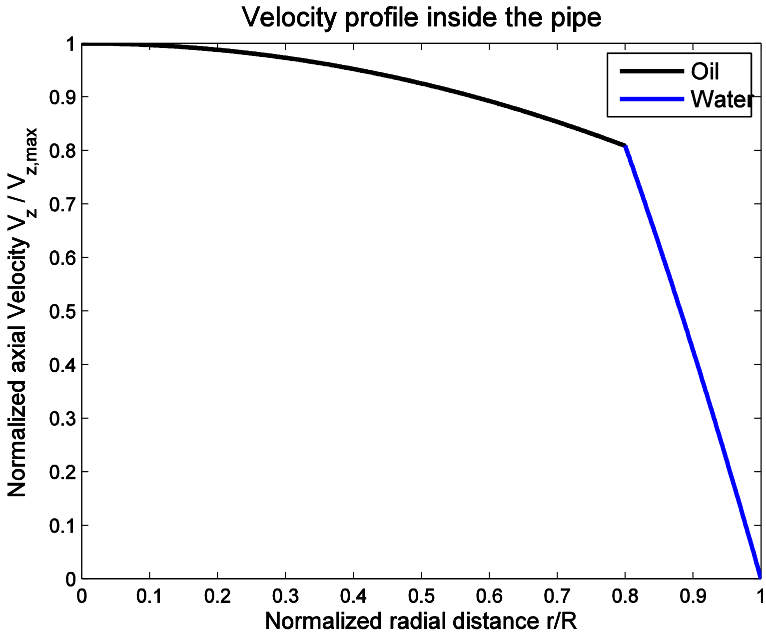

Figure 1. Normalized Velocity profile with $R_1 = 0.8R$ and $\mu_1 = 7.5\mu_2$.

In Figure 1 it is plotted the normalized velocity profile inside the pipe. It may seem that the shear stress at the interface is not the same for both fluids (different slopes), but it is important to remember that there are two different viscosities. This can be seen in the normalized shear stress profile in figure 2.

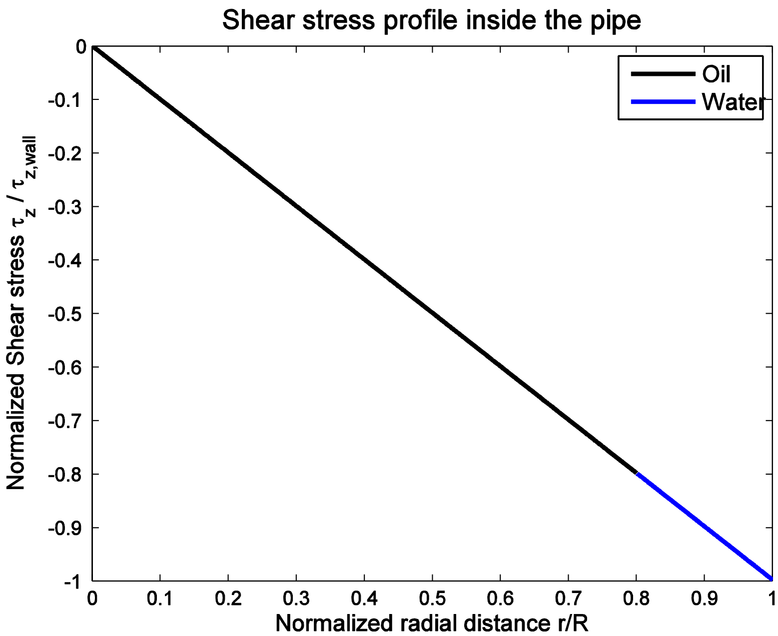

Figure 2. Normalized shear stress profile with $R_1 = 0.8R$ and $\mu_1 = 7.5\mu_2$.

1. f) In order to obtain the volume flow rates for both regions, the velocity profile (7) must be integrated within the appropriate intervals.

For the water region $(R_1 \leq r < R)$:

$$Q_w = \int_{R_1}^{R} \int_0^{2\pi} \frac{\Delta p}{4\mu_2 L} (R^2 - r^2)\, r d\theta dr$$

$$\boxed{Q_w = \frac{\Delta p\, \pi}{8\mu_2 L} \left(R^2 - {R_1}^2\right)^2} \tag{9}$$

For the oil region $(0 < r < R_1)$:

$$Q_o = \int_0^{R_1} \int_0^{2\pi} \frac{\Delta p}{4L} \left(\frac{R^2 - R_1^2}{\mu_2} + \frac{R_1^2 - r^2}{\mu_1} \right) r d\theta dr$$

$$\boxed{Q_o = \frac{\Delta p\, \pi}{2L} \left(\frac{R_1^2 R^2 - R_1^4}{2\mu_2} + \frac{R_1^4}{4\mu_1} \right)} \tag{10}$$

Exercise 2·

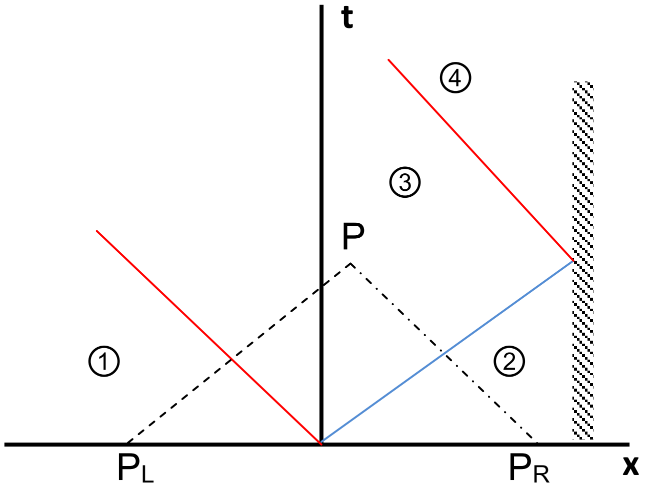

Figure 3. xt diagram.

The xt diagram is plotted in figure 3, where the red lines represent compression waves and blue line expansion waves. The dashed and dot-dashed lines represent the characteristic lines among which Riemann invariants are constant.

- In region ① the wave has not arrived yet and thus conserves its initial state of u=0; $p=p_1$.
- In region ② the wave has not arrived yet and thus conserves its initial state of u=0; $p=p_0$.
- In region ③ the wave has passed and velocity and pressure are to be computed.
- In region ④ the reflected wave has passed and velocity and pressure reach their final state of u=0; $p=p_1$. This is because the characteristics lines that go through a point P in this region go back to region ①.

Knowing that Riemann invariants remain constant among the characteristic lines, we can compute the values of p and u in a point P in region ③, retrieving information from points P_L and P_R.

Along characteristic line $x - a_o t = cte$ (Dashed line in figure 3):

$$\frac{u}{a_0} + \frac{1}{\gamma}\frac{p}{p_o} = cte \rightarrow \frac{u(P)}{a_0} + \frac{1}{\gamma}\frac{p(P)}{p_o} = \frac{u(P_L)}{a_0} + \frac{1}{\gamma}\frac{p(P_L)}{p_o}$$

$$\frac{u(P)}{a_0} + \frac{1}{\gamma}\frac{p(P)}{p_o} = \frac{0}{a_0} + \frac{1}{\gamma}\frac{p_1}{p_o} \tag{11}$$

Along characteristic line $x + a_o t = cte$(Dot-dashed line in figure 3):

$$\frac{u}{a_0} - \frac{1}{\gamma}\frac{p}{p_o} = cte \rightarrow \frac{u(P)}{a_0} - \frac{1}{\gamma}\frac{p(P)}{p_o} = \frac{u(P_R)}{a_0} - \frac{1}{\gamma}\frac{p(P_R)}{p_o}$$

$$\frac{u(P)}{a_0} - \frac{1}{\gamma}\frac{p(P)}{p_o} = \frac{0}{a_0} - \frac{1}{\gamma}\frac{p_0}{p_o} \tag{12}$$

Computing (11) - (12):

$$\frac{2}{\gamma}\frac{p(P)}{p_o} = \frac{1}{\gamma}\left(\frac{p_1}{p_o} + \frac{p_0}{p_o}\right)$$

We obtain:

$$\boxed{p(P) = \frac{p_1 + p_0}{2}} \tag{13}$$

And finally computing (11) + (12):

$$\frac{2u(P)}{a_0} = \frac{1}{\gamma}\left(\frac{p_1}{p_o} - \frac{p_0}{p_o}\right)$$

We obtain:

$$\boxed{u(P) = \frac{a_0}{2\gamma}\left(\frac{p_1}{p_0} - 1\right)} \tag{14}$$