AFM-Homework4

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Solution 1

Part a

According to definition of this stream function, this flow is irrotational. Since $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\psi = Ur^2\sin(2\theta) = 2Ur^2\sin\theta\cos\theta = 2Uxy$$

So by definition of ψ in terms of x and y,

$$u = \frac{\partial\psi}{\partial y} = 2Ux, \quad v = -\frac{\partial\psi}{\partial x} = -2Uy, \quad \mathbf{v} = \sqrt{u^2+v^2} = 2U\sqrt{x^2+y^2}$$

In the stagnation point $(x, y) = (0, 0)$, $(u, v) = (0, 0)$ and far from this point $x = \infty$ or $y = \infty$, the velocity is equal to ∞.

To obtain an expression for the pressure, we recall Bernoulli's equation,

$$\int_1^2 \frac{\partial\mathbf{v}}{\partial t}.\, ds + \frac{1}{2}\mathbf{v}_2^2 + \frac{p_2}{\rho} + gz_2 = \frac{1}{2}\mathbf{v}_1^2 + \frac{p_1}{\rho} + gz_1$$

Point 1 is the stagnation point and point 2 is an arbitrary point in flow, so $\mathbf{v}_1 = 0$. we do not have body force, so $gz_1 = gz_2 = 0$. The integral term should be zero because flow is steady,

$$\frac{1}{2}(4U^2(x^2+y^2)) + \frac{p_2}{\rho} = \frac{p_1}{\rho} \Rightarrow p_2 = p_1 - 2\rho U^2(x^2+y^2)$$

Part b

The Navier-Stokes equations are as follow,

$$\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\\
&\rho\Big(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\Big) = -\frac{\partial p}{\partial x} + \mu\nabla^2 u + \rho b_x\\
&\rho\Big(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\Big) = -\frac{\partial p}{\partial y} + \mu\nabla^2 v + \rho b_y
\end{aligned}$$

By eliminating zero terms in Navier-Stokes equations, we get

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ \rho u \frac{\partial u}{\partial x} &= -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} \\ \rho v \frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial y^2} \end{aligned} \tag{1}$$

according to part a, $u = 2Ux$, $v = -2Uy$, $\frac{\partial u}{\partial x} = 2U$, $\frac{\partial v}{\partial y} = -2U$, $\frac{\partial p}{\partial x} = -4\rho U^2 x$, $\frac{\partial p}{\partial y} = -4\rho U^2 y$, and $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y^2} = 0$. Therefore, equations (1) hold.

Since x axis Plane is without motion, boundary condition for viscous problem is non-slip condition, so u and v, components of velocity field should be zero close to x axis. This means $u(x, 0) = v(x, 0) = 0$, but $u(x, 0) = 2Ux \neq 0$. So, non-slip boundary condition does not hold.

Part c

By assuming $u = 2Uxf'(y)$, the continuity equation (the first equation in (1)) gives,

$$\frac{\partial v}{\partial y} = -2Uf'(y) \Rightarrow v = -2Uf(y)$$

and boundary condition is $u(x, 0) = v(x, 0) = 0$, so $f'(0) = 0$ and $f(0) = 0$. Besides, when $y \to \infty$, $f(y) \to y$ based on condition for potential flow in part a.

Part d

The y-momentum equation is,

$$4\rho U^2 f(y)f'(y) = -\frac{\partial p}{\partial y} - 2\mu U f''(y)$$

by integrating with respect to y, we get an expression for the pressure as following:

$$p(x, y) = -2\rho U^2 (f(y))^2 - 2\mu U f'(y) + h(x) \tag{2}$$

In (2), we use this fact that $(f^2)' = 2ff'$. Moreover, since pressure is a function of both x and y, the constant of this integration should depends on x. To find $h(x)$, as we point out in part c, for large y, we have $f(y) \to y$ and $f'(y) \to 1$. So the equation (2) is,

$$p(x, y) \to -2\rho U^2 y^2 - 2\mu U + h(x)$$

by recovering the pressure as in part a,

$$
\begin{aligned}
& p_1 - 2\rho U^2(x^2+y^2) = -2\rho U^2 y^2 - 2\mu U + h(x) \quad \text{(for large } y\text{)} \\
& \Rightarrow h(x) = p_1 + 2\mu U - 2\rho U^2 x^2
\end{aligned}
$$

Thus, the pressure distribution is,

$$
\begin{aligned}
p(x,y) &= p_1 + 2\mu U - 2\rho U^2 x^2 - 2\rho U^2 (f(y))^2 - 2\mu U f'(y) \\
&= p_1 + 2\mu U(1 - f'(y)) - 2\rho U^2 (x^2 + (f(y))^2)
\end{aligned}
$$

Part e

By pressure distribution in hand, we rewrite the x-momentum equation as follow:

$$
4\rho U^2 x (f')^2 - 4\rho U^2 x f f'' = 4\rho U^2 x + 2\mu U x f'''
$$

This is an ordinary differential equation of order 3 and depends on only y. So, we divide both sides of this equation by $4\rho U^2 x$,

$$
(f')^2 - f f'' = 1 + \frac{\nu}{2U} f'''
$$

this equation is a nonlinear ordinary differential equation of order 3 with boundary condition as follow:

$$
f(0) = 0, \quad f'(0) = 0, \quad f(y) \to y, \text{ and } f'(y) \to 1 \text{ as } y \to \infty
$$

By changing variable as follow:

$$
\eta = \sqrt{\frac{2U}{\nu}} y, \quad g(\eta) = \sqrt{\frac{2U}{\nu}} f(y)
$$

we have,

$$
f'(y) = g'(\eta), \quad f''(y) = \sqrt{\frac{2U}{\nu}} g''(\eta), \quad f'''(y) = \frac{2U}{\nu} g'''(\eta)
$$

therefore, the differential equation is,

$$
g''' + g g'' - (g')^2 + 1 = 0
$$

where,

$$
g(0) = g'(0) = 0, \quad g'(\eta) \to \eta \text{ as } \eta \to \infty
$$

This ODE is almost similar to boundary layer equation (PDE) with the same boundary condition. So, we can solve it numerically.

Solution 2

By assumption of the problem, the velocity profile is,

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\frac{y^2}{\delta^2}$$

where the boundary conditions are,

$$\begin{aligned} &u = 0 \quad \text{at } y = 0 \\ &u = U, \ \frac{\partial u}{\partial y} = 0 \quad \text{at } y = \delta \end{aligned}$$

So we have,

$$\begin{aligned} &a = 0 \\ &1 = b + c \\ &\frac{1}{U}\frac{\partial u}{\partial y} = \frac{b}{\delta} + \frac{2cy}{\delta^2}\Big|_{y=\delta} \Rightarrow 0 = b + 2c \end{aligned}$$

so, $b = 2$, $c = -1$, and we can write velocity profile as follow:

$$u = U\Big(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\Big)$$

Now we can compute displacement thickness and momentum thickness. In Kármán-Pohlhausen approximation, we have

$$\begin{aligned} \int_0^\delta (U-u)u\,dy &= U^2\int_0^\delta \Big(1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\Big)\Big(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\Big)\,dy \\ &= U^2\int_0^\delta \Big(\frac{2y}{\delta} - \frac{5y^2}{\delta^2} + \frac{4y^3}{\delta^3} - \frac{y^4}{\delta^4}\Big)\,dy \\ &= U^2\Big(\frac{y^2}{\delta} - \frac{5y^2}{3\delta^2} + \frac{y^4}{\delta^3} - \frac{y^5}{5\delta^4}\Big)_0^\delta \\ &= U^2\big(\delta - \frac{5}{3}\delta + \delta - \frac{\delta}{5}\big) = \frac{2}{15}U^2\delta \end{aligned}$$

also we have,

$$\frac{\tau_0}{\rho} = \nu\Big(\frac{\partial u}{\partial y}\Big)_0 = \nu\Big(\frac{2U}{\delta} - \frac{2Uy}{\delta^2}\Big)_{y=0} = \frac{2U\nu}{\delta}$$

thus,

$$\frac{2}{15}U^2\frac{d\delta}{dx} = \frac{2U\nu}{\delta}, \quad \delta(0) = 0$$

By solving this differential equation, we get

$$\frac{\delta^2}{2} = \frac{15\nu}{U}x + C, \quad C = 0 \quad \Rightarrow \delta = 5.4772\sqrt{\frac{\nu x}{U}}$$

Moreover,

$$\frac{\tau_0}{(1/2)\rho U^2} = \frac{4U\nu}{\delta U^2} = \frac{4\nu}{(5.4772)U\sqrt{\frac{\nu x}{U}}} = \frac{0.7303}{\sqrt{Re}}$$

So,

$$\frac{\delta}{x} = \frac{5.4772}{\sqrt{Re}}, \quad \frac{\theta}{x} = \frac{0.7303}{\sqrt{Re}}$$

Note that the Blasius solution is,

$$\frac{\delta}{x} = \frac{5}{\sqrt{Re}}, \quad \frac{\theta}{x} = \frac{0.664}{\sqrt{Re}}$$

and the one which is obtained by cubic velocity profile is,

$$\frac{\delta}{x} = \frac{4.64}{\sqrt{Re}}, \quad \frac{\theta}{x} = \frac{0.646}{\sqrt{Re}}$$

By comparing our solution and cubic velocity, we find the solution obtained cubic velocity profile is better approximation and is closer to Blasius solution which is exact solution. however, our approximation can be good because the difference between it and Blasius solution is less than 0.1