ADVANCED FLUID MECHANICS

HOMEWORK (4)

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(1) a) $\psi(r,\theta) = U r^2 \sin(2\theta)$

- Velocity field in polar coordinates

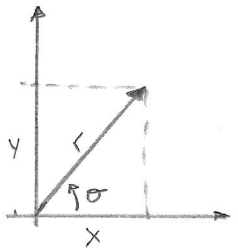
$$u_r = \frac{1}{r} \cdot \frac{\partial\psi}{\partial\theta} = 2Ur\cos(2\theta)$$

$$u_\theta = \frac{\partial\psi}{\partial r} = 2Ur\sin(2\theta)$$

- Velocity field in cartesian coordinates

$$u_x = u_r \cdot \cos\theta - u_\theta \cdot \sin\theta$$

$$u_y = u_r \cdot \sin\theta + u_\theta \cdot \cos\theta$$



$x = r.\cos\theta \;/\; \cos\theta = x/r = x/\sqrt{x^2+y^2}$

$y = r.\sin\theta \;/\; \sin\theta = y/r = y/\sqrt{x^2+y^2}$

$r^2 = x^2 + y^2$

$$u_x = 2U\underbrace{r\cos(2\theta).\cos(\theta)}_{x} - 2U\underbrace{r\sin(2\theta).\sin(\theta)}_{y}$$

$$u_y = 2U\overbrace{r\cos(2\theta).\sin(\theta)}^{y} + 2U\overbrace{r\sin(2\theta).\cos(\theta)}^{x}$$

$$u_x = 2U\cos(2\theta).x - 2U\sin(2\theta)\,y$$

$$u_y = 2U\sin(2\theta)\,x + 2U\cos(2\theta)\,y$$

- Trigonometric identities

$$\cos(2\theta) = 2.\cos^2\theta - 1$$

$$\sin(2\theta) = 2.\operatorname{sen}\theta..\cos\theta$$

$$u_x = 2U\left(2\cdot\cos^2(\theta)-1\right)x - 2U\left(2\cdot\sin(\theta)\cdot\cos(\theta)\right)y$$

$$u_x = -2Ux + 4U\cos(\theta)\left(\cos(\theta)x + \sin(\theta)y\right)$$

$$u_x = -2Ux + 4U\frac{x}{\sqrt{x^2+y^2}}\left(\frac{x^2}{\sqrt{x^2+y^2}} + \frac{y^2}{\sqrt{x^2+y^2}}\right)$$

$$u_x = -2Ux + 4U\left(\frac{x\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}}\right) = 2Ux$$

$$u_y = 2U\left(2\cdot\sin(\theta)\cdot\cos(\theta)\right)x + 2U\left(2\cos^2(\theta)-1\right)y$$

$$u_y = -2Uy + 4U\cos\theta\left(\cos(\theta)y - \sin(\theta)x\right)$$

$$u_y = -2Uy + 4U\frac{x}{\sqrt{x^2+y^2}}\underbrace{\cancel{\left(\frac{xy}{\sqrt{x^2+y^2}} - \frac{xy}{\sqrt{x^2+y^2}}\right)}}_{0} = -2Uy$$

$$\boxed{\begin{aligned} u_x &= 2.U.x \\ u_y &= -2.U.y \end{aligned}}$$

"Velocity vector field in cartesian coordinates"

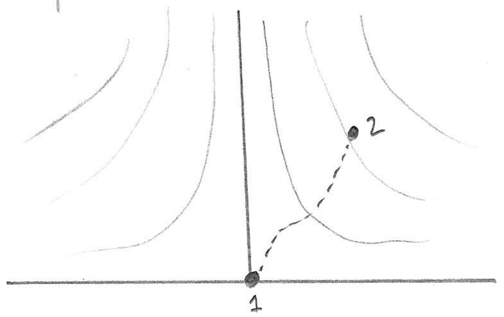
<u>Boundary conditions</u> → $v\cdot n = 0$ at surfaces

- $u_x = 0$ at $x=0$ ✓
- $u_y = 0$ at $y=0$ ✓

the flow is irrotational → Bernoulli's eq. between any 2 points

$$\frac{\partial u_x}{\partial y} = 0 \qquad \frac{\partial u_y}{\partial x} = 0 \quad \rightarrow \omega = 0$$

$$\underbrace{\cancel{\int_1^2 \frac{\partial u}{\partial t}\, ds}}_{\text{STATIONARY}} + \left(\frac{1}{2}u_2^2 + \frac{P_2}{\rho} + \underbrace{\cancel{F_2}}_{=0}\right) - \left(\underbrace{\cancel{\frac{1}{2}u_1}}_{=0} + \frac{P_1}{\rho} + \underbrace{\cancel{F_1}}_{=0}\right) = 0$$



$$u_2^2 = 4U^2(x^2+y^2)$$

$F_1 = F_2$
e plane flow

$$2U^2(x^2+y^2) + \frac{P_2 - P_1}{\varphi} = 0 \qquad P_1 = P_{max} \text{ (stagnation point)}$$

$$\boxed{P(x,y) = P_{max} - 2\varphi U^2(x^2+y^2)}$$

b) Navier-Stokes eq

- Mass cons. → $\nabla \cdot u = 0$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 2U - 2U = 0 \checkmark$$

- Momentum cons. (x)

$$\varphi\left(\cancel{\frac{\partial u_x}{\partial t}}^{=0} + u_x \frac{\partial u_x}{\partial x} + u_y \cancel{\frac{\partial u_x}{\partial y}}^{=0}\right) = -\frac{\partial p}{\partial x} + \mu \nabla^2 u_x + \cancel{\varphi g_x}^{=0}$$

$$\varphi \cdot 2Ux \, 2U = 2U^2\varphi(2x) + \mu \cdot (0+0+0)$$

$$\varphi 4U^2 x = \varphi 4U^2 x \checkmark$$

- Momentum cons (y)

$$\varphi\left(\cancel{\frac{\partial u_y}{\partial t}} + u_x \cancel{\frac{\partial u_y}{\partial x}} + u_y \frac{\partial u_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \cancel{\mu \nabla^2 u_y} + \cancel{\varphi g_y}$$

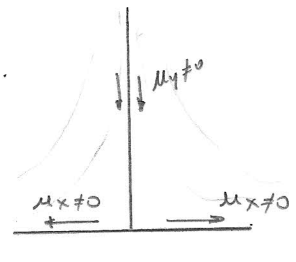
$$\varphi \cdot (-2Uy)(-2U) = 2U^2\varphi(2y)$$

$$4U^2\varphi y = 4U^2\varphi y \checkmark$$

the velocity and pressure fields verify N-S. equations.

BC in viscous problem → no slip condition.

$$\left.\begin{array}{l} u(x=0)=0 \begin{cases} u_x = 0 \\ u_y = -2Uy \neq 0 \end{cases} \\ \\ u(y=0)=0 \begin{cases} u_x = 2Ux \neq 0 \\ u_y = 0 \end{cases} \end{array}\right\} \text{doesn't fullfill BC!}$$



c) $u_x = 2Ux\, f'(y)$

$u_y = -2U f(y)$

BC for function f

at $y=0 \rightarrow u_x = u_y = 0$

far from the boundary $\rightarrow u_x = 2Ux$
$u_y = -2Uy$

$2Ux \cdot f'(0) = 0 \rightarrow f'(0)=0$

$-2Uf(0) = 0 \rightarrow f(0)=0$

$2Ux = 2Ux\, f'(y) \rightarrow f'(y)=1$

$-2Uy = -2U f(y) \rightarrow \boxed{f(y)=Y}$

d) $\varphi\left(\cancel{\frac{\partial u_y}{\partial t}} + \cancel{u_x \frac{\partial u_x}{\partial x}} + u_y \frac{\partial u_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu \nabla^2 v + \cancel{\varphi g_y}$

$\varphi(-2U f(y))\cdot(-2U f'(y)) = -\frac{\partial p}{\partial y} + \mu\left(0 - 2U f''(y)\right)$

$4\varphi U^2 f(y) f'(y) = -\frac{\partial p}{\partial y} - 2U\mu f''(y)$ $\qquad \nu = \frac{\mu}{\varphi}$

$\frac{\partial p}{\partial y} \cdot \frac{1}{\varphi} = -2U\nu f''(y) - 4\varphi U^2 f(y) f'(y)$

Integrating.

$P(x,y) = -2U\mu \overset{1}{f'(y)} - 2\varphi U^2 \overset{Y}{f^2(y)} + P_0(x)$

for $y \approx \infty \rightarrow$ P must be the same as inviscid

$P(x,y) = P_{max} - 2\varphi U^2 (x^2+y^2)$

$-2U\mu - 2\varphi U^2 y^2 + P_0(x) = P_{max} - 2\varphi U^2(x^2+y^2)$

$$P_0 = P_{max} + 2\mu U - 2\varphi U^2 x^2 = P_{max} + 2U(\mu - U\varphi x^2)$$

$$P(x,y) = -2U\mu f'(y) - 2\varphi U^2 f^2(y) + P_{max} + 2U(\mu - U\varphi x^2)$$

e)

$$\varphi\left(\cancel{\frac{\partial u_x}{\partial t}} + u_x \frac{\partial u_x}{\partial x} + u_y \cdot \frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu \nabla^2 u_x + \cancel{\varphi b_x}$$

$$\varphi(2Ux f'(y) \cdot 2Uf'(y) - 2Uf(y) \cdot 2Ux f''(y)) = 4U^2\varphi x + 2\mu Ux f'''(x)$$

$$\frac{4U^2\varphi x ([f'(y)]^2 - f(y)f''(y))}{4U^2\varphi x} = \frac{4U^2\varphi x}{4U^2\varphi x} + \frac{2\mu Ux f'''(x)}{4U^2\varphi x}$$

$$(f'(y))^2 - f(y) f''(y) = 1 + \frac{\nu f'''(x)}{2U}$$

$$\frac{\nu}{2U} \cdot f'''(y) + f(y) f''(y) - (f'(y))^2 + 1 = 0$$

from point (c) we know the BC.

at $y=0$

$$\frac{\nu}{2U} \cdot f'''(0) + \underbrace{f(0)}_{=0} f''(0) - \cancel{\underbrace{f'(0)}_{=0}}^2 + 1 = 0 \rightarrow f'''(0) = -\frac{2U}{\nu}$$

at $y \neq 0$

$$\frac{\nu}{2U} f'''(y) + f(y) f''(y) - f'(y)^2 + 1 = 0$$

$$\frac{\nu}{2U} f'''(y) + \cancel{y \cdot 0} - \cancel{1^2} \cancel{+1} = 0$$

$$\frac{\nu}{2U} f'''(y) = 0 \rightarrow \boxed{f''(y) = 0}$$

2) - Blasius exact solution $\frac{\delta}{x} = \frac{5}{\sqrt{Re}}$, $\frac{\theta}{x} = \frac{0,664}{\sqrt{Re}}$

- Cubic Profile . $\frac{\delta}{x} = \frac{4,64}{\sqrt{Re}}$, $\frac{\theta}{x} = \frac{0,646}{\sqrt{Re}}$

→ Quadratic Profile:

$$\begin{cases} \frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2 \\ u = 0 \quad \text{at} \quad y = 0 \quad (1) \\ u = U \quad \text{at} \quad y = \delta \quad (2) \\ \frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = \delta \quad (3) \end{cases}$$

- Applying BC

(1)

$$0 = a + \frac{b.0}{\delta} + c\left(\frac{0}{\delta}\right)^2 \rightsquigarrow a = 0$$

(2)

$$\frac{\cancel{U}}{\cancel{U}} = \frac{b\cancel{\delta}}{\cancel{\delta}} + c\left(\frac{\cancel{\delta}}{\cancel{\delta}}\right)^2 \rightsquigarrow b + c = 1 \quad \therefore \quad b = 1 - c$$

(3)

$$\frac{\partial u}{\partial y} = \left(\frac{b}{\delta} + 2c\left(\frac{y}{\delta}\right)\cdot\frac{1}{\delta}\right).U \rightsquigarrow 0 = \left(\frac{b}{\delta} + \frac{2c}{\delta}\right).U$$

$$b + 2c = 0$$

$$1 - c + 2c = 0$$

$$\boxed{c = -1}$$

$$\boxed{b = 2}$$

$$\frac{u}{U} = \frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2$$

$$\frac{\partial u}{\partial y} = \frac{2U}{\delta}\left(1 - \frac{y}{\delta}\right)$$

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For a flat Plate $\frac{d}{dx}(U^2\theta) = \frac{\tau_0}{\rho}$

$$U^2\theta = \int_0^\delta u(U-u)\,dy \rightarrow \frac{d}{dx}\int_0^\delta u(U-u)\,dy = \frac{\tau_0}{\rho}$$

(5) (4)

(4)

$$\frac{\tau_0}{\rho} = \nu\left(\frac{\partial u}{\partial y}\right)\Big|_0 = \nu\left(\frac{2U}{\delta}\right)$$

(5)

$$\frac{d}{dx}\int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)U\left(U - U\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\right)dy$$

$$U^2\frac{d}{dx}\int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right)dy$$

$$U^2\frac{d}{dx}\int_0^\delta \left(\frac{2y}{\delta} - \frac{5y^2}{\delta^2} + \frac{4y^3}{\delta^3} - \frac{y^4}{\delta^4}\right)dy \rightarrow U^2\frac{d}{dx}\left[\frac{y^2}{\delta} - \frac{5y^3}{3\delta^2} + \frac{y^4}{\delta^3} - \frac{y^5}{5\delta^4}\right]_0^\delta$$

$$\rightarrow U^2\frac{d}{dx}\left(\frac{2\delta}{15}\right)$$

(5) = (4)

$$U^{\cancel{2}}\frac{d}{dx}\left(\frac{\cancel{2}\delta}{15}\right) = \frac{\cancel{2}\cancel{U}\nu}{\delta}$$

$$\rightarrow \boxed{\frac{d\delta}{dx} = \frac{15\nu}{U\delta}}$$

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$$\frac{d\delta}{dx} = \frac{15\nu}{U\delta} \rightarrow \int d\delta \cdot \delta = \int \frac{15\nu}{U} dx$$

$$\frac{\delta^2}{2} = \frac{15\nu}{U} \cdot x + C \rightarrow \delta = \sqrt{\frac{30\nu}{U} \cdot x + 2C}$$

$\rightarrow$ at $x = 0 \rightarrow \delta = 0$

$$0 = \sqrt{\frac{30 \cdot \nu}{U} \cdot 0 + 2C} \rightarrow \boxed{C = 0}$$

$$\delta = 5{,}477 \sqrt{\frac{\nu x}{U}} \rightarrow \text{Adimensionalize}$$

$$\frac{\delta}{x} = \frac{5{,}477}{x} \sqrt{\frac{\nu x}{U}} \rightarrow \frac{\delta}{x} = \frac{5{,}477}{\sqrt{x}\sqrt{x}} \sqrt{\frac{\nu x}{U}}$$

$$\rightarrow \frac{\delta}{x} = 5{,}477 \sqrt{\frac{\nu \cancel{x}}{U \cdot x^{2}}} \rightarrow \boxed{\frac{\delta}{x} = \frac{5{,}477}{\sqrt{Re}}}$$

Obtaining θ

$$U^2\theta = \int_0^\delta u(U-u)\,dy \rightarrow U^2\theta = \left[\frac{y^2}{\delta} - \frac{5y^3}{3\delta^2} + \frac{y^4}{\delta^3} - \frac{y^5}{\delta^4} \right]_0^\delta U^2$$

$$\theta = \frac{2}{15}\delta \rightarrow \frac{\theta}{x} = \frac{5{,}477 \cdot 2}{15\sqrt{Re}}$$

$$\boxed{\frac{\theta}{x} = \frac{0{,}729}{\sqrt{Re}}}$$

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- As expected, the cubic aproximation gives a better result than the quadratic when comparing with the exact solution.