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Homework 4: Navier-Stokes equations and Boundary Layer

Problem 1)

Part a)

The stream functions can be expressed in Cartesian coordinates:

$$\psi = Ur^2\sin(2\theta) = U(x^2+y^2)\sin\left(2\arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right) = 2\frac{U(x^2+y^2)y}{x^2+y^2}\sqrt{x^2+y^2-y^2} = 2Uxy \qquad (1)$$

The velocity field is:

$$u = \frac{\partial\psi}{\partial y} = 2Ux \qquad (2)$$

$$v = -\frac{\partial\psi}{\partial x} = -2Uy \qquad (3)$$

This velocity field satisfies the boundary conditions for the velocity:

- Wall: At y=0 there is no normal velocity to the wall → $v(x, y=0) = 0$
- Stagnation point: At x=0,y=0 the velocity is 0 → $\begin{cases} u(x=0, y=0) = 0 \\ v(x=0, y=0) = 0 \end{cases}$

Moreover:

- For $y > 0 \rightarrow \begin{cases} v < 0 \\ \text{As } y \rightarrow 0, v \rightarrow 0 \end{cases}$
- For $x > 0 \rightarrow \begin{cases} u > 0 \\ \text{As } x \rightarrow 0, u \rightarrow 0 \end{cases}$
- For $x < 0 \rightarrow \begin{cases} u < 0 \\ \text{As } x \rightarrow 0, u \rightarrow 0 \end{cases}$
- At x=0 there is a symmetry axis.

The pressure field can be obtained using Bernoulli's equation. Since the fluid is assumed to be inviscid and incompressible, and the velocity field obtained is irrotational, Bernoulli's equation is valid along any line.

For pressure distribution, Bernoulli equation is applied between stagnation point and an arbitrary point in the flow field:

$$\frac{1}{2}\left(\sqrt{u(x,y)^2+v(x,y)^2}\right)^2 + \frac{p(x,y)}{\rho} = \frac{p_0}{\rho} \qquad (4)$$

Substituting the expressions for the velocity:

$$p(x,y) = p_0 - 2\rho U^2(x^2+y^2) \qquad (5)$$

Part b)

The Navier-Stokes equations are satisfied:

- Continuity equation:

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial \mathbf{y}} = 2\mathrm{U} - 2\mathrm{U} = \mathbf{0} \tag{6}$$

- x-momentum:

$$\boldsymbol{\rho}\left(\mathbf{u}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \mathbf{v}\frac{\partial \mathbf{u}}{\partial \mathbf{y}}\right) = -\frac{\partial \mathbf{p}}{\partial \mathbf{x}} + \boldsymbol{\mu}\boldsymbol{\nabla}^2\mathbf{u} \tag{7}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = 4\rho U^2 x$$

$$-\frac{\partial p}{\partial x} + \mu\nabla^2 u = 4\rho U^2 x$$

- y-momentum:

$$\boldsymbol{\rho}\left(\mathbf{u}\frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{y}}\right) = -\frac{\partial \mathbf{p}}{\partial \mathbf{y}} + \boldsymbol{\mu}\boldsymbol{\nabla}^2\mathbf{v} \tag{8}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = 4\rho U^2 y$$

$$-\frac{\partial p}{\partial y} + \mu\nabla^2 v = 4\rho U^2 y$$

In the Navier-Stokes equations, the boundary condition at the wall is a non-slip condition:

$$\begin{cases} u(x, y = 0) = 0 \\ v(x, y = 0) = 0 \end{cases}$$

Using the velocity-field obtained in a), the non-slip boundary condition is only satisfied at the stagnation point:

$$\begin{cases} u(x, y = 0) = 2Ux \\ v(x, y = 0) = 0 \end{cases}$$

Part c)

Horizontal velocity for the viscous problem is as follow

$$u = 2Uxf'(y) \tag{9}$$

Using continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2Uf'(y) + \frac{\partial v}{\partial y} = 0 \rightarrow \frac{\partial v}{\partial y} = -2Uf'(y)$$

Integrating:

$$v = -2Uf(y) + C$$

When $x = 0, C = 0$, hence without any loss of generality we can write as follow,

$$v = -2Uf(y) \tag{10}$$

The function f(y) must be chosen so that the non-slip boundary condition at the wall is satisfied and the irrotational flow obtained in a) is recovered for $y \to \infty$:

- At y=0:

$$\begin{cases} u(x, y = 0) = 2Uxf'(0) = 0 \\ v(x, y = 0) = -2Uf(0) = 0 \end{cases} \to \begin{cases} f'(0) = 0 \\ f(0) = 0 \end{cases}$$

- At $y \to \infty$:

$$\begin{cases} u(x, y \to \infty) = 2Uxf'(y \to \infty) = 2Ux \\ v(x, y \to \infty) = -2Uf(y \to \infty) = -2Uy \end{cases} \to \begin{cases} f'(y \to \infty) = 1 \\ f(y \to \infty) = y \end{cases}$$

Part d)

Momentum equation along y axis (8) can be used to obtain an expression for the pressure:

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = 4\rho U^2 f(y)f'(y) \tag{11}$$

$$\mu\nabla^2 v = \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = -2\mu U f''(y) \tag{12}$$

Using (8), (11) and (12):

$$\frac{\partial p}{\partial y} = -2U\big(\mu f''(y) + 2\rho U f(y) f'(y)\big) \tag{13}$$

Integrating (13):

$$p(x, y) = p_0 + p(x) - 2U\mu f'(y) - 2\rho U^2 \big(f(y)\big)^2 \tag{14}$$

For $y \to \infty$, the pressure distribution obtained for the inviscid problem (5) must be recovered:

$$p_0 - 2\rho U^2(x^2 + y^2) = p_0 + p(x) - 2U\mu f'(y \to \infty) - 2\rho U^2\big(f(y \to \infty)\big)^2 \tag{15}$$

For $y \to \infty, \begin{cases} f(y) = y \\ f'(y) = 1 \end{cases}$

So:

$$p(x) = 2U\mu - 2\rho U^2 x^2 \tag{16}$$

Adding (16) in (14):

$$p(x, y) = p_0 + 2U\mu\big(1 - f'(y)\big) - 2\rho U^2\left[x^2 + \big(f(y)\big)^2\right] \tag{17}$$

Part e)

Differentiating equation **(** 17**)** with respect to x, we get:

$$\frac{\partial p}{\partial x} = -4\rho U^2 x \tag{18}$$

Substituting **(** 18**)** and the expressions of u and v in the x-momentum equation (7) we get the following,

$$4U^2 x\rho\left(\left(f'(y)\right)^2 - f(y)f''(y)\right) = 4\rho U^2 x + 2Ux\mu f'''(y) \tag{19}$$

Thus:

$$\left(f'(y)\right)^2 - f(y)f''(y) = 1 + \frac{\mu}{2U\rho}f'''(y) \tag{20}$$

The boundary conditions for f were stated in c). In order to be able to solve the ODE:

- At y=0:

$$\begin{cases} f'(0) = 0 \\ f(0) = 0 \end{cases}$$

- At y → ∞:

$$f'(y \to \infty) = 1$$

Equation **(** 20**)** is a nonlinear third order ordinary differential equation and can be conveniently solved numerically to obtain the expression for *f(y)* using the boundary conditions stated before. In post processing, *f(y)* can be used to estimate the velocity field and other parameters of interest such as shear strain τ , drag force F_D etc.

2)

The Karman momentum integral equation is:

$$\frac{d}{dx}(U^2\theta) + \delta^* U \frac{dU}{dx} = \frac{\tau_0}{\rho} \tag{21}$$

Where θ is the momentum thickness:

$$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \tag{22}$$

In a uniform flow over a flat:

$$\frac{dU}{dx} = 0 \tag{23}$$

Using (21), (22) and (23):

$$\frac{d}{dx}\int_0^\infty u(U-u)dy = \frac{\tau_0}{\rho} \tag{24}$$

We assume a parabolic profile:

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2 \tag{25}$$

The boundary conditions that must be satisfied are:

1. At $y = 0 \rightarrow u = 0$
2. At $y = \delta \rightarrow u = U$
3. At $y = \delta \rightarrow \frac{\partial u}{\partial y} = 0$

From condition 1:

$$a = 0$$

Using condition 2:

$$\frac{u}{U}(y = \delta) = a + b\frac{\delta}{\delta} + c\left(\frac{\delta}{\delta}\right)^2 = b + c = 1$$

From condition 3:

$$\left.\frac{\partial u}{\partial y}\right|_{y=\delta} = \frac{bU}{\delta} + \frac{2cU}{\delta} = 0$$

Thus:

$$\begin{cases} a = 0 \\ b = 2 \\ c = -1 \end{cases}$$

$$\frac{u}{U}(y) = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2 \tag{26}$$

Now we can compute:

- Shear stress τ_0

$$\tau_0 = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{2U}{\delta}\mu \tag{27}$$

- Boundary layer thickness δ:

Using (24), (27) and solving the integral in the left hand side of (24):

$$\delta d\delta = 15\frac{\mu}{U\rho}\, dx \tag{28}$$

Integrating (28):

$$\delta = \sqrt{\frac{30\mu x}{U\rho}} \tag{29}$$

Definning the Reynolds number as:

$$Re = \frac{Ux\rho}{\mu} \tag{30}$$

$$\frac{\delta}{x} = \sqrt{30}\frac{1}{\sqrt{Re}} \approx \frac{5.477}{\sqrt{Re}} \tag{31}$$

- Momentum thickness θ:

Using (22) and (26):

$$\frac{\theta}{x} \approx \frac{0.730}{\sqrt{Re}} \tag{32}$$

Comparing the results obtained with the ones obtained for both the Blasius "exact" solution and with the ones obtained assuming a cubic velocity profile:

	Cuadratic function	Cubic function	Blasius
$\frac{u}{U}$	$2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$	$\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$	
$\frac{\delta}{x}$	$\frac{5.477}{\sqrt{Re}}$	$\frac{4.64}{\sqrt{Re}}$	$\frac{5}{\sqrt{Re}}$
$\frac{\theta}{x}$	$\frac{0.730}{\sqrt{Re}}$	$\frac{0.646}{\sqrt{Re}}$	$\frac{0.664}{\sqrt{Re}}$

In order to compare the different solutions, in Fig. 1 $\frac{u}{U}$ is plotted vs $\eta = \frac{y}{\sqrt{\frac{\mu x}{\rho U}}}$. If we consider the Blasius approximation as the "exact" solution, the error made in both the cubic and cuadratic approximations can be obtained when comparing with Blasius solution (Fig. 2). As can be seen in both figures, the cubic approximation is better than the cuadratic approximation.

Cubic approximation results to be a better approximation than the cuadratic as it incorporates the boundary condition $\frac{d^2u}{dy^2} = 0$ for y=0. Hence if this condition is applied along with no slip boundary condition in x direction momentum equation at y = 0, we retrive our original assumption i.e. $\frac{dp}{dx} = 0$. On the other hand, using quadratic approximation, x direction momentum equation at y = 0 gives the following:

$$\frac{dp}{dx} = \frac{-2U}{\delta^2}$$

This result goes contrary to our assumption that the pressure is uniform within the domain. Therefore, cubic velocity profile is a better approximation of the 'exact' Blasius solution.

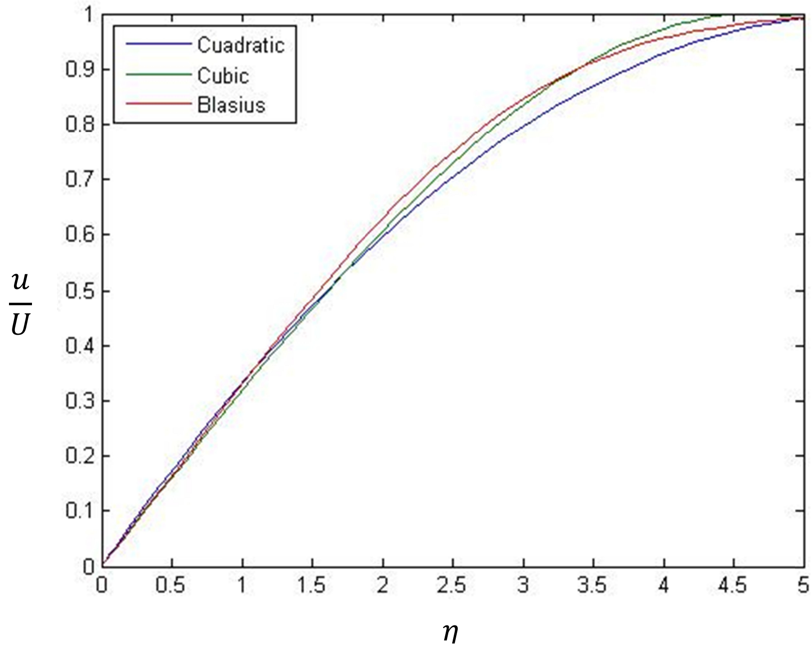
Cuadratic
Cubic
Blasius
$\frac{u}{U}$
η

Fig. 1

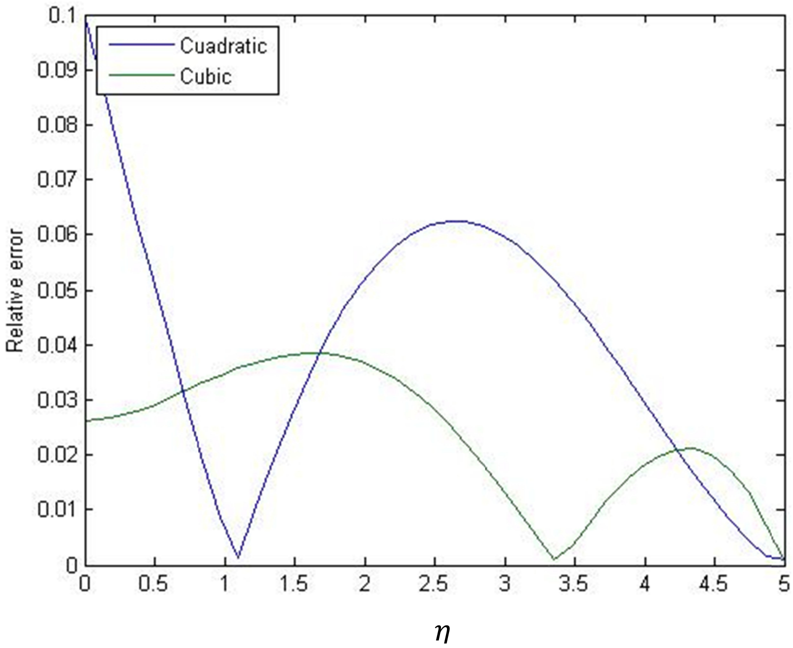
Cuadratic
Cubic
Relative error
η

Fig. 2