ADVANCED FLUID MECHANICS

HOMEWORK ASSIGNMENT 2

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ASSIGNMENT No. 2

① <u>Assumptions</u>

Flow is considered as (i) constant density
(ii) inviscid.
(iii) Steady

(a) <u>Velocity field</u>

$$V_r = \begin{cases} C_1 r & r \le a \\ C_2 / r & r \ge a \end{cases}$$

Applying mass balance. for $r > a$.

$$Q = \pi a^2 V = 2\pi h V_r$$

$$V_r = a^2 V / 2hr \Rightarrow C_2 = \frac{a^2 V}{2h}.$$

When $r = a$. mass balance is

$$C_1 a = \frac{a^2 V}{2ha} \Rightarrow C_1 = \frac{V}{2h}.$$

$$\boxed{a^2 C_1 = C_2}$$

(b) <u>Estimating pressure field</u>

Applying Bernoulli at PQR stream line.

$$P_{\text{Stag-nation}} = p_0 + \frac{1}{2}\rho V^2 = p(r) + \frac{1}{2}\rho V(r)^2$$

So

$$p(r) - p_0 = \cancel{p(r)} + \frac{1}{2}\rho\left[V_r^2 - V_0^2\right]$$

when $r \le a$

$$p(r) - p_0 = \frac{1}{2}\rho V^2\left[1 - r^2/4h^2\right]$$

when $r \ge a$

$$p(r) - p_0 = \frac{1}{2}\rho V^2\left[1 - a^4/4h^2r^2\right].$$

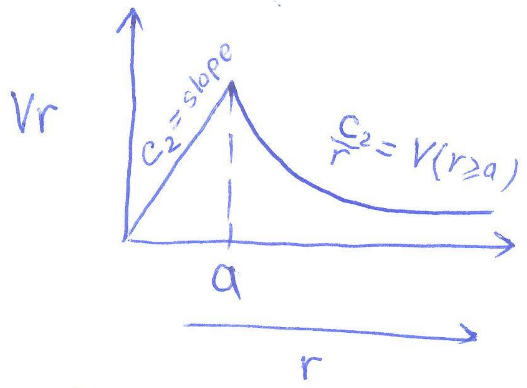
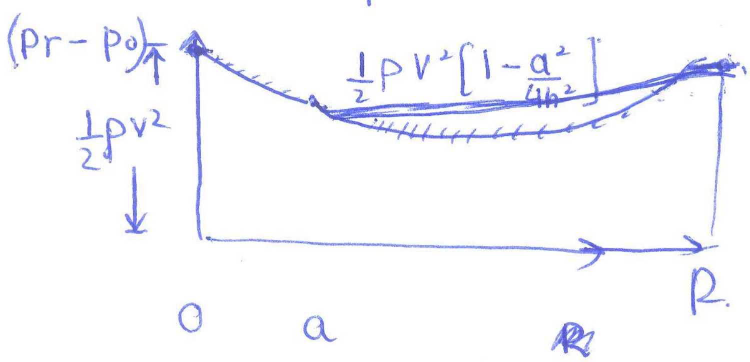


Fig1: Velocity plot



$$\frac{1}{2}\rho v^2\left[1 - \frac{a^4}{4h^2R^2}\right]$$

As $R \to \infty$

$$\Rightarrow P_{(r)} - P(0) = \frac{1}{2}\rho v^2$$

Fig2: Pressure plot

(c) Force on Disc

Total force is integration of Gauge pressure from $0 \to R$.

$$F_T = \int_0^R P(r)\, 2\pi r\, dr = \int_0^a \pi\rho V^2\left[r - \frac{r^3}{4h^2}\right]dr +$$

$$\pi\rho V^2 \int_a^R \left[r - \frac{a^4}{4h^2 r}\right] dr$$

$$F_T = \frac{1}{2}\pi\rho v^2 \left[r^2 - \frac{r^4}{8h^2} \right]_0^a +$$

$$\frac{1}{2}\pi\rho v^2 \left[\left| r^2 \right|_a^R - \left| \frac{a^4}{2h^2} \ln(r) \right|_a^R \right].$$

$$= \frac{1}{2}\pi\rho v^2 \left[a^2 - \frac{a^4}{8h^2} + R^2 - a^2 - \frac{a^4}{2h^2} \ln\left(\frac{R}{a}\right) \right].$$

$$F_T = \frac{1}{2}\pi\rho v^2 \left[R^2 - \frac{a^4}{8h^2} - \frac{a^4}{2h^2} \ln\left(\frac{R}{a}\right) \right]$$

$$F_T = \frac{1}{2}\pi\rho v^2 \left[R^2 - \frac{a^4}{8h^2} + \frac{a^4}{2h^2} \ln\left(\frac{a}{R}\right) \right]$$

In Abore expression, first term is also positire, Last term is always negative because $R > a$, Combination of term 1 and 2 will be negative when

$$R^2 - \frac{a^4}{8h^2} < 0.$$

$$\cancel{\frac{8R^2}{a}} \quad \cancel{< h^2} \quad \cancel{h >} \quad \cancel{\frac{8R^2}{a^2}}$$

$$R^2 < \frac{a^4}{8h^2}$$

$$\boxed{h < \frac{a^2}{2\sqrt{2}\, h}}$$

Under given condition

$F_T = -W$

$\rho_{air} = 1.2\,kg/m^3$

$a = 10^{-2}m$. $\quad \rho = 1.2\,kg/m^3$.

$R = 5\,E{-2}m$

$h = 0.1E{-2}m$

$W = mg = (10E{-3})(9.8) = 98E{-3}.N$

plugging these values in the following equation.

$$-W = \frac{1}{2}\pi\rho V^2\left[R^2 - \frac{a^4}{8h^2} + \frac{a^4}{2h^2}\left(\frac{a}{R}\right)\right].$$

$V = 2.78\,m/s$

(d) <u>Unsteady flow condition</u>

$$\cancel{\frac{\partial M_{sys}}{\partial t} = \dot{m}_{out} + \dot{m}_{in}}$$

$\int \rho(v - v_c)\cdot n\,dA$

$$\cancel{0 = \frac{d}{dt}(\pi r^2 h) - \pi r^2 V + 2\pi r h V_r} = 0$$

m_1

$\cancel{\rho \pi r^2 h(t)}\ \dot{m}_{out} + \dot{m}_{in} = \frac{dM}{dt}$

$$\frac{dM}{dT} = \rho V_r 2\pi r h - \rho V a^2 \pi + \rho \pi r^2 \dot{h}(t) = 0$$

$$\boxed{V_r = \frac{V a^2}{2rh} - \frac{r^2 \dot{h}(t)}{2rh}}$$

This expression is only valid $r > a$ but for $r < a$ due to the volume

For $r > a$

To obtain C_1

$$\frac{V a^2 - a^2 \dot{h}}{2ah} = C_1$$

$$\boxed{C_1 = \frac{V - \dot{h}(t)}{2h}}$$

$$V_r = \begin{cases} \dfrac{(V - \dot{h})r}{2h} & r < a \\[2ex] \dfrac{V a^2 - r^2 \dot{h}(t)}{2rh} & r > a \end{cases}$$

To obtain the expression for $h(t)$

$$\int^{R} \frac{dV}{dt}\, ds + \frac{P_o}{\rho} = \frac{P_p}{\rho} + \frac{V^2}{2} \qquad P_p = P_0 + \frac{\rho r^2}{2}$$

$p(t)$ +

Applying Bernoulli from P to R

$$\int_P^R \frac{dV}{dt} \cdot ds + \frac{P_o}{\rho} + \frac{V_R^2}{2} = \frac{P_P}{\rho}$$

TO obtain P_P we apply Bernoulli again

$$P_P = P_o + \frac{\rho V^2}{2}$$

$$\int_P^R \frac{dV}{dt} \cdot ds + \frac{P_o}{\rho} + \frac{V_R^2}{2} = \frac{P_o}{\rho} + \frac{V^2}{2}$$

$$\int_O^P \frac{dV}{dt} \cdot ds + \int_P^R \frac{dV}{dt} ds + \frac{P_o}{\rho} + \frac{V_R^2}{2}$$

$$= \frac{P_o}{\rho} + \frac{V^2}{2}$$

$$\int_O^a \frac{dV}{dt} dr + \cancel{\frac{P_o}{\rho}} + \frac{V_R^2}{2} = \cancel{\frac{P_o}{\rho}} + \frac{V^2}{2}$$

$$+ \int_a^R \frac{dV}{dt} dr$$

$$\int_O^a \frac{d}{dt}\left(\frac{V_r - \dot{h}(t) r}{2}\right) dr + \int_a^R \frac{d}{dt}\left(\frac{Va^2 - r^2 \dot{h}(t)}{2rh}\right) dr$$

$$+ \cancel{\frac{P}{\rho}} \quad \frac{V_R^2}{2} = \frac{V^2}{2}$$

$$\int_0^a \frac{d}{dt}\left(\frac{V_0 r - \dot{h}(t)\, r}{2}\right) dr + \int_a^R \frac{d}{dt}\left(\frac{V a^2 - r^2 \dot{h}(t)}{2 r h}\right) dr$$

$$+ \frac{1}{2}\left[\frac{V a^2 - R^2 \dot{h}(t)}{2 R h}\right]^2 - \frac{V^2}{2} = 0$$

(2) Solution

In ~~Laplacian~~ cylindrical polar coordinate we have the following expression for ~~gradient~~ Laplacian of scalar field

$$\nabla^2 a = \frac{\partial^2 a}{\partial r^2} + \frac{1}{r}\frac{\partial a}{\partial r} + \frac{1}{r^2}\frac{\partial^2 a}{\partial \theta^2}$$

given $\psi(r,\theta) = f(r)\sin\theta$ with $f(r) = r^{\alpha}$.

$$\psi(r,\theta) = r^{\alpha}\sin\theta$$

$$\nabla^2\psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}$$

$$\nabla^2\psi = (\alpha^2-\alpha) r^{\alpha-2}\sin\theta + \frac{1}{r}\alpha r^{\alpha-1}\sin\theta - r^{\alpha-2}\sin\theta$$

$$\nabla^2\psi = r^{\alpha-2}\sin\theta\,(\alpha^2-1)$$

For flow to be irrotational we have $\nabla^2\psi = 0$

$$\alpha = \pm 1$$

So our streamline function is

$$\psi = a\frac{\sin\theta}{r} + b\,r\sin\theta$$

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$V_r = \frac{1}{r}\left(\frac{a_1 \cos\theta}{r} + b\, r\cos\theta\right)$$

$$\boxed{V_r = \frac{a_1 \cos\theta}{r^2} + b\cos\theta}$$

$$V_\theta = -\frac{\partial \psi}{\partial r}$$

$$= -\left(-a_1 \frac{\sin\theta}{r^2} + b\sin\theta\right)$$

$$\boxed{V_\theta = \frac{a_1 \sin\theta}{r^2} - b\sin\theta}$$

Applying Boundary conditions now

$$r \to \infty$$

$$V_r = b\cos\theta$$

$$V_\theta = -b\sin\theta$$

$$U_x = \sqrt{V_r^2 + V_\theta^2} = b$$

$$\boxed{U_x = b}$$

Second Boundary condition

at $r = R$, $\theta = \pi$ we have $V_r = 0$

$V_\theta = 0$

$$V_r = \frac{+a_1 \cos\pi}{R^2} + b\cos\pi$$

$$V_r = -\frac{a_1}{R^2} - U_x$$

$$0 = -\frac{a_1}{R^2} - U_x$$

$$\boxed{a_1 = -U_x R^2}$$

$$\psi = -U_x R^2 \frac{\sin\theta}{r} + U_x r \sin\theta$$

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = -U_x \frac{R^2}{r^2}\cos\theta + U_x \cos\theta$$

$$\boxed{V_r = U_x\left[1 - \frac{R^2}{r^2}\right]\cos\theta}$$

$$V_\theta = -\frac{\partial \psi}{\partial x} = U_x \frac{R^2}{r^2}\sin\theta - U_x \sin\theta$$

$$\boxed{V_\theta = -U_x \sin\theta\left[1 + \frac{R^2}{r^2}\right]}$$

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At $r = R$ we have the following Boundary conditions

$V_r = 0$, $V_\theta = -2U_x \sin\theta$

our stream line function ~~for~~ is

$$U = \sqrt{\left(\frac{1}{r}\frac{d\psi}{d\theta}\right)^2 + \left(\frac{\partial\psi}{\partial r}\right)^2} = U_x \quad @ \quad \begin{array}{l} r = R \\ \theta = \pi \\ V_r = 0 \\ V_\theta = 0 \end{array}$$

(c) Velocity field

$$\cancel{U} = \sqrt{V_r^2 + V_\theta^2}$$

$$\cancel{U} = \sqrt{4U_x^2 \sin^2\theta}$$

$$V = \sqrt{4U_x^2 \sin^2\theta}$$

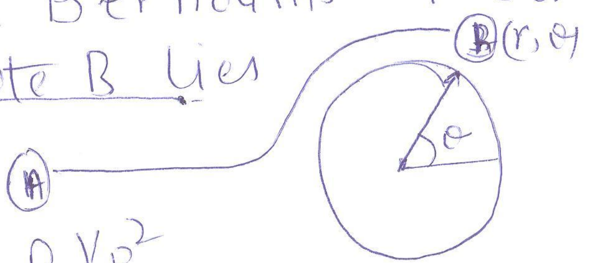
$$\boxed{V = 2U_x \sin\theta}$$

@ $r = R$

$V_r = 0$

$V_\theta = -2U_x \sin\theta$

(d) Applying the Bernoullis theorem on streamline $\overline{AB}$ (Note B lies on the cylinder surface)



$$P_A + \frac{1}{2}\rho V_A^2 = P_2 + \frac{1}{2}\rho V_B^2$$

$$P_1 + \frac{1}{2}\rho U_x^2 = P_2 + \frac{1}{2}\rho(2U_x \sin\theta)^2$$

$$P_1 - P_2 = \frac{1}{4}\rho U^2\left[2\sin^2\theta - \frac{1}{2}\right]$$

$$P_1 - P_2 = U^2\rho\left[\cos 2\theta - \frac{1}{2}\right]$$

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NET FORCE ACTING ON CYLINDER

$$F_H = \int_0^{2\pi} P R \cos\theta \, d\theta, \quad F_V = \int_0^{2\pi} P R \sin\theta \, d\theta$$

$$F_H = \int_0^{2\pi} u^2 \rho \left[\cos 2\theta - \frac{1}{2}\right] R \cos\theta \, d\theta$$

$$= u^2 \rho R \left[\int_0^{2\pi} \cos 2\theta \cos\theta \, d\theta - \frac{1}{2} \int_0^{2\pi} \cos\theta \, d\theta \right]$$

We will integrate $\int_0^{2\pi} \cos 2\theta \cos\theta \, d\theta$ now

$$\int_0^{2\pi} \cos\theta \cos 2\theta \, d\theta = \cos\theta \int_0^{2\pi} \cos 2\theta \, d\theta - \left[\sin\theta \int_0^{2\pi} \sin 2\theta \, d\theta - \int_0^{2\pi} \frac{\cos 2\theta \cos\theta}{2} \, d\theta \right]$$

~~Let~~

~~Now we will integrate~~ $\cancel{\int_0^{2\pi} \cos 2\theta \cos\theta}$

Let $A = \int_0^{2\pi} \frac{\cos 2\theta \cos\theta \, d\theta}{2}$

$$\cancel{= \cos\theta \int_0^{2\pi} \cos 2\theta \, d\theta - \left[\sin\theta \int_0^{2\pi} \sin 2\theta \right]}$$

$$= \cos\theta \int_0^{2\pi} \cos 2\theta \, d\theta - \left[\sin\theta \int_0^{2\pi} \sin 2\theta \, d\theta - A \right.$$

$$A = \cos\theta \int_0^{2\pi} \cos 2\theta \, d\theta - \sin\theta \int_0^{2\pi} \sin 2\theta \, d\theta$$

$$A = \left[\frac{\cos\theta \sin 2\theta}{2} \right]_0^{2\pi} + \left[\frac{\sin\theta \cos 2\theta}{2} \right]_0^{2\pi} = 0$$

$$F_H = \rho U^2 R \left[-\frac{1}{2} \sin\theta \right]_0^{2\pi}$$

$$\boxed{F_H = 0}$$

$$F_V = \int_{-\pi}^{\pi} U^2 \rho R \left[\cos 2\theta - \frac{1}{2} \right] \sin\theta \, d\theta$$

$$= U^2 \rho R \quad - \int_{\pi}^{0} \left[\cos 2\theta - \frac{1}{2} \right] \sin(-\theta) \, d\theta \; +$$

$$\int_0^{\pi} \left(\cos 2\theta - \frac{1}{2} \right) \sin\theta \, d\theta$$

Side Note

$$\left[\begin{array}{l} \sin(-\theta) = -\sin\theta \\ \cos(-\theta) = \cos\theta \end{array} \right]$$

$$= U^2 R \rho \left[\int_0^{\pi} \left(\cos 2\theta - \frac{1}{2} \right)(-\sin\theta) + \int_0^{\pi} \left[\cos\theta - \frac{1}{2} \right] \sin\theta \, d\theta \right.$$

$$= U^2 \rho R \left[- \int_0^{\pi} \left(\cos\theta - \frac{1}{2} \right) \sin\theta + \int_0^{\pi} \left(\cos 2\theta - \frac{1}{2} \right) \sin\theta \, d\theta \right]$$

$$F_V = 0$$

So net force $F = \sqrt{F_V^2 + F_H^2}$

$$\Rightarrow F = 0$$

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<u>Discussion</u>

In practice we should get a drag force but in this case as we have solved stream lines for incompressible flow, ~~in~~ so there would be no viscosity in the constitutive relation and pressure on both side of cylinder is even so our drag will be equal to 0