ADVANCED FLUID MECHANICS HomeWork 1

Paris Dilip Mulye

November 3, 2015

- Problem 1 (a)

  Left Hand Side = $\nabla \cdot (\nabla \times F)$

  $= \nabla \cdot (\epsilon_{ijk} F_{k,j})$

  $= (\epsilon_{ijk} F_{k,j})_{,i}$

  $= \epsilon_{ijk} F_{k,ji}$

  $= \epsilon_{123} F_{3,21} + \epsilon_{132} F_{2,31} + \epsilon_{213} F_{3,12} + \epsilon_{231} F_{1,32} + \epsilon_{312} F_{2,13} + \epsilon_{321} F_{1,23}$

  (Rest all terms are zero because ϵ will be zero if i=j or i=k or j=k)

  $= F_{3,21} - F_{3,12} - F_{2,31} + F_{2,13} - F_{1,32} + F_{1,23}$

  $= 0 =$ Right Hand Side

- Problem 1 (b)

  Left Hand Side = $\nabla \times (\nabla \times F) = \nabla \times \epsilon_{ijk} F_{k,j}$

  $= \epsilon_{pqi} (\epsilon_{ijk} F_{k,j})_{,q} = \epsilon_{pqi} \epsilon_{ijk} F_{k,jq}$

  $= \epsilon_{pqi} \epsilon_{jki} F_{k,jq} = (\delta_{pj} \delta_{qk} - \delta_{pk} \delta_{qj}) F_{k,jq}$

  $= \delta_{pj} \delta_{qk} F_{k,jq} - \delta_{pk} \delta_{qj} F_{k,jq} = \delta_{pj} F_{q,jq} - \delta_{pk} F_{k,jj}$

  $= F_{q,pq} - F_{p,jj} = F_{q,qp} - F_{p,jj}$

  $= \nabla(\nabla \cdot F) - \nabla^2 F =$ Right Hand Side

- Problem 1 (c)

  Left Hand Side = $\nabla \cdot (F \times G)$

  $= \nabla \cdot (\epsilon_{ijk} F_i G_j)$

  $= (\epsilon_{ijk} F_i G_j)_{,k}$

  $= \epsilon_{ijk} F_{i,k} G_j + \epsilon_{ijk} F_i G_{j,k}$

  $= \epsilon_{jki} F_{i,k} G_j - \epsilon_{ikj} G_{j,k} F_i$

  $= (\nabla \times F) \cdot G - (\nabla \times G) \cdot F$

  = Right Hand Side

- Problem 2

Using Maxwell's equation of Energy,

$de - Tds + pdV = 0$

But specific volume $V = \frac{1}{\rho}$ making $dV = -\frac{1}{\rho^2}d\rho$

Hence, the equation becomes,

$de - Tds - \frac{p}{\rho^2}d\rho = 0$

Taking Material Derivatives,

$\frac{De}{Dt} - T\frac{Ds}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = 0$

Multiplying every side by ρ

$\rho\frac{De}{Dt} - T\rho\frac{Ds}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt} = 0$...........(1)

From energy conservation equation which states,

$\rho\frac{De}{Dt} = \sigma : \nabla v - \nabla \cdot q$...........(2)

Given a Newtonian fluid, $\sigma = -pI + \lambda(\nabla \cdot v)I + 2\mu\nabla^S v$

Taking a double dot product with ∇v

$\sigma : \nabla v = -pI : \nabla v + \lambda(\nabla \cdot v)I : \nabla v + 2\mu\nabla^S v : \nabla v$

$\sigma : \nabla v = -p(\nabla \cdot v) + \lambda(\nabla \cdot v)^2 + 2\mu(\nabla^s v) : \nabla v$

Using $K = \lambda + \frac{2}{3}\mu$, $\lambda = K - \frac{2}{3}\mu$

$\sigma : \nabla v = -p(\nabla \cdot v) + \lambda(\nabla \cdot v)^2 + 2\mu(\nabla^s v) : \nabla v$

$\sigma : \nabla v = -p(\nabla \cdot v) + K(\nabla \cdot v)^2 - \frac{2}{3}\mu(\nabla \cdot v)^2 + 2\mu(\nabla^s v) : \nabla v$

Substituting this result in (2) and in (1),

$-p(\nabla \cdot v) + K(\nabla \cdot v)^2 - \frac{2}{3}\mu(\nabla \cdot v)^2 + 2\mu(\nabla^s v) : \nabla v - \nabla \cdot q - T\rho\frac{Ds}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt} = 0$

$-\frac{p}{\rho}\left(\frac{D\rho}{Dt} + \rho(\nabla \cdot v)\right) + K(\nabla \cdot v)^2 - \frac{2}{3}\mu(\nabla \cdot v)^2 + 2\mu(\nabla^s v) : \nabla v - \nabla \cdot q - T\rho\frac{Ds}{Dt} = 0$

Using continuity equation, $\frac{D\rho}{Dt} + \rho(\nabla \cdot v) = 0$

$K(\nabla \cdot v)^2 - \frac{2}{3}\mu(\nabla \cdot v)^2 + 2\mu(\nabla^s v) : \nabla v - \nabla \cdot q - T\rho\frac{Ds}{Dt} = 0$

Dividing throughout by T,

$\frac{K}{T}(\nabla \cdot v)^2 - \frac{2}{3}\frac{\mu}{T}(\nabla \cdot v)^2 + 2\frac{\mu}{T}(\nabla^s v) : \nabla v - \frac{\nabla \cdot q}{T} - \rho\frac{Ds}{Dt} = 0$.........(3)

Integrating over Material Volume V_t,

$$\int_{V_t} \frac{K}{T}(\nabla \cdot v)^2 dV = \phi_1$$

$$\int_{V_t} -\frac{2}{3}\frac{\mu}{T}(\nabla \cdot v)^2 dV = \phi_2$$

$$\int_{V_t} +2\frac{\mu}{T}(\nabla^s v) : \nabla v dV = \phi_3$$

$$\int_{V_t} -\frac{\nabla \cdot q}{T} dV = \phi_4$$

$$\int_{V_t} -\rho \frac{Ds}{Dt} dV = \phi_5$$

Such that, $\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 = 0$

ϕ_1 Calculation:

Since K ≥ 0 is given, T is always positive and $(\nabla \cdot v)^2 \geq 0$, $\phi_1 \geq 0$

ϕ_5 Calculation:

using Reynold's Transport Theorem,

$$\frac{D}{Dt}\int_{V_t} \rho s dV = \int_{V_t} \frac{D}{Dt}(\rho s) dV + \int_{S_t} (\rho s v) \cdot (n) dS$$

$$\frac{D}{Dt}\int_{V_t} \rho s dV = \int_{V_t} \rho \frac{Ds}{Dt} dV + \int_{V_t} s \frac{D\rho}{Dt} dV + \int_{V_t} \rho s (\nabla \cdot v) dV$$

$$\frac{D}{Dt}\int_{V_t} \rho s dV = \int_{V_t} \rho \frac{Ds}{Dt} dV + \int_{V_t} s \left(\frac{D\rho}{Dt} + \rho(\nabla \cdot v)\right) dV$$

using continuity equation,

$$\frac{D}{Dt}\int_{V_t} \rho s dV = \int_{V_t} \rho \frac{Ds}{Dt} dV$$

Thus, $\phi_5 = -\frac{D}{Dt}\int_{V_t} \rho s dV$

ϕ_4 Calculation:

$$\nabla \cdot (q/T) = \left(\frac{q_i}{T}\right)_{,i} = \frac{q_{i,i}}{T} - \frac{q_i T_{,i}}{T^2} = \frac{\nabla \cdot q}{T} - \frac{q \cdot \nabla T}{T^2}$$

Given, $q = -k\nabla T$,

Thus, $\phi_4 = \int_{V_t} -\nabla \cdot \left(\frac{q}{T}\right) dV + \int_{V_t} \frac{k \nabla T \cdot \nabla T}{T^2} dV$

$\phi_2 + \phi_3$ Calculation:

$$\phi_2 + \phi_3 = \int_{V_t} \frac{\mu}{T}\left(-\frac{2}{3}(\nabla \cdot v)^2 + 2(\nabla^s v) : \nabla v\right) dV$$

Working on the term inside the integral,

$$\left(-\frac{2}{3}(\nabla \cdot v)^2 + 2(\nabla^s v) : \nabla v\right)$$

After expanding the terms component wise and combining terms,

$$\left(\frac{4}{3}(v_{1,1}^2 + v_{2,2}^2 + v_{3,3}^2 - v_{1,1}v_{2,2} - v_{2,2}v_{3,3} - v_{1,1}v_{3,3}\right) +$$
$$\left(v_{1,2}^2 + v_{1,3}^2 + v_{2,1}^2 + v_{2,3}^2 + v_{3,1}^2 + v_{3,2}^2\right) + 2\left(v_{1,2}v_{2,1} + v_{1,3}v_{3,1} + v_{2,3}v_{3,2}\right)$$

which can be converted into sum of squares like this,

$$\frac{2}{3}\left[(v_{1,1} - v_{2,2})^2 + (v_{2,2} - v_{3,3})^2 + (v_{3,3} - v_{1,1})^2\right] + \left[(v_{1,2} + v_{2,1})^2 + (v_{1,3} + v_{3,1})^2 + (v_{2,3} + v_{3,2})^2\right]$$

Thus, $\phi_2 + \phi_3 = \int_{V_t} \frac{\mu}{T} A dV$ where A ≥ 0

Since, μ is given to be positive and T is always positive, $\phi_2 + \phi_3 \geq 0$

Main Equation:

Moving back to the main equation involving ϕ and substituting the relation for ϕ_4,

$$\phi_1 + (\phi_2 + \phi_3) + \int\limits_{V_t} k\frac{(\nabla T)^2}{T^2} dV = -\phi_5 + \int\limits_{V_t} \nabla \cdot \left(\frac{q}{T}\right) dV$$

$$\phi_1 + (\phi_2 + \phi_3) + \int\limits_{V_t} k\frac{(\nabla T)^2}{T^2} dV = \frac{D}{Dt}\int\limits_{V_t} \rho s dV + \int\limits_{V_t} \nabla \cdot \left(\frac{q}{T}\right) dV$$

Using divergence theorem,

$$\phi_1 + (\phi_2 + \phi_3) + \int\limits_{V_t} k\frac{(\nabla T)^2}{T^2} dV = \frac{D}{Dt}\int\limits_{V_t} \rho s dV + \int\limits_{S_t} \left(\frac{q \cdot n}{T}\right) dS$$

Since $\phi_1 \geq 0, \phi_2 + \phi_3 \geq 0$

and k is given to be positive and $\frac{(\nabla T)^2}{T^2}$ will be always $\geq= 0$,

All terms on the left hand side are $\geq= 0$

$$\frac{D}{Dt}\int\limits_{V_t} \rho s dV + \int\limits_{S_t} \left(\frac{q \cdot n}{T}\right) dS \geq 0$$

$$\frac{D}{Dt}\int\limits_{V_t} \rho s dV \geq -\int\limits_{S_t} \left(\frac{q \cdot n}{T}\right) dS$$

This proves the inequality.