Advanced Fluid Mechanics

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Computational Mechanics

Assignment 2

1.) Assuming a control volume, cylindrical in shape with radius r.

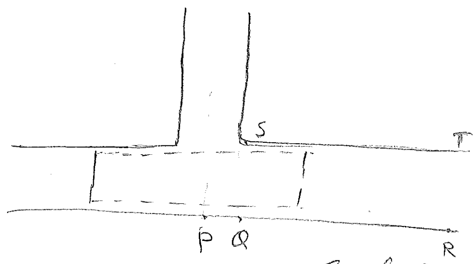


Fig (1a)

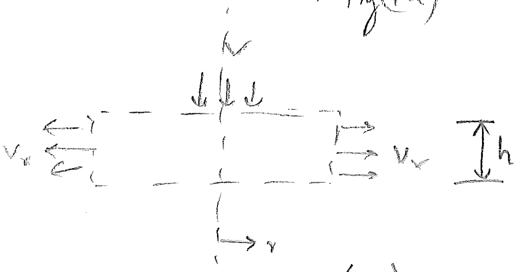


Fig (1b)

For a steady state flow & fixed control volume, we have

$$\oint \rho \vec{v} (\vec{v} \cdot \hat{n}) ds = 0$$

$$\Rightarrow \quad -\rho V \pi a^2 + \rho V_r \cdot 2\pi r h = 0$$

Using $V_r = C_2/r \quad r > a$, we have

$$\Rightarrow \quad C_2 = \frac{Va^2}{2h}$$

$$\therefore \quad V_r = \frac{Va^2}{2rh} \qquad r > a$$

$\therefore \quad V_r = C_1 r \quad \forall r < a$, ~~we balance the~~ we have

$$V_r\Big|_{r=a} = \frac{C_2}{r}\Big|_{r=a} = C_1 r\Big|_{r=a}$$

$$\Rightarrow \quad \frac{Va^2}{2ah} = C_1 a \Rightarrow \quad C_1 = \frac{V}{2h}$$

$$\therefore \quad V_r = \begin{cases} \dfrac{Vr}{2h} & r < a \\ \dfrac{Va^2}{2rh} & r \geq a \end{cases}$$

1.b) Applying Bernoulli's equation across 1 & 2_r

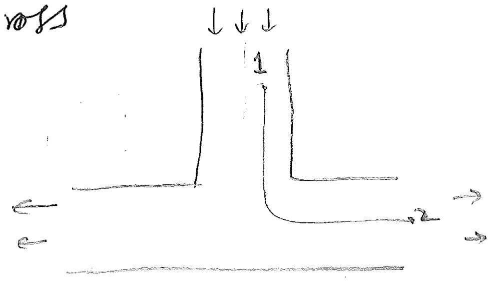


Fig 2

$$p_0 + \frac{1}{2}\rho V^2 = p(r) + \frac{1}{2}\rho v_r^2$$

$$\Rightarrow p(r) - p_0 = \frac{1}{2}\rho V^2 - \frac{1}{2}\rho v_r^2$$

$$= \begin{cases} \frac{1}{2}\rho V^2\left(1 - \frac{r^2}{4h^2}\right) & r < a \\ \frac{1}{2}\rho V^2\left(1 - \left(\frac{a^2}{2rh}\right)^2\right) & r \geq a \end{cases}$$

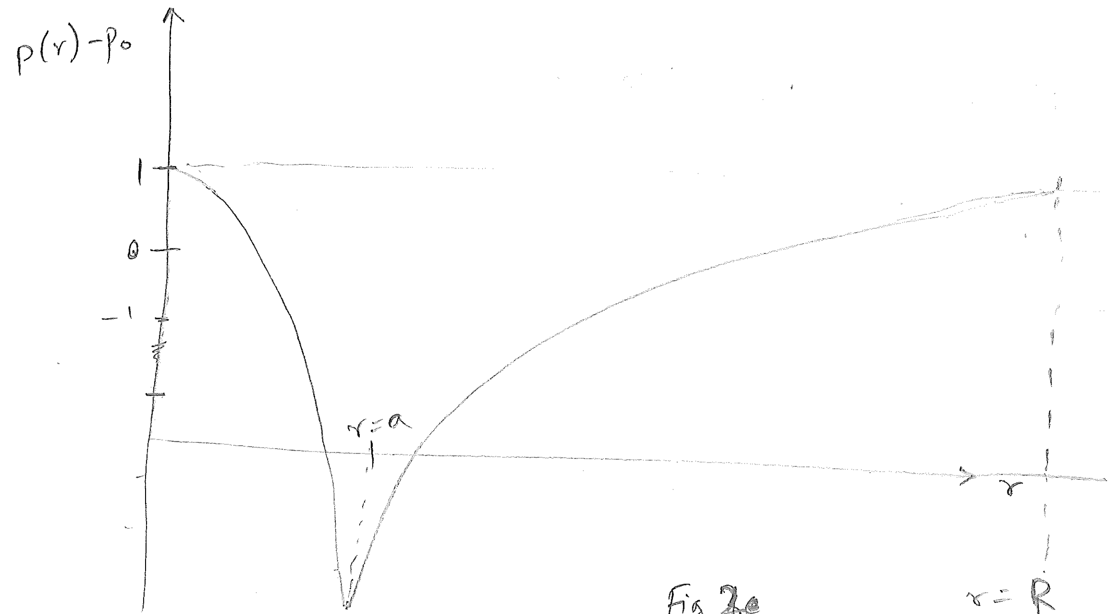


Fig 3a

$$F_p = \cancel{\int_0^R p(r)\, dr\, 2\pi r\, dr} \int_0^R (p(r) - p_0)\, 2\pi r\, dr$$

$$= \int_0^a \frac{1}{2}\rho V^2\left(1 - \frac{r^2}{4h^2}\right) 2\pi r\, dr + \int_a^R \frac{1}{2}\rho V^2\left(1 - \frac{a^4}{4h^2}\cdot\frac{1}{r^2}\right) 2\pi r\, dr$$

$$\cancel{= \frac{1}{2}\rho V^2 \cdot \pi R^2 - \frac{1}{2}\rho V^2 \int_0^a r^2}$$

$$= \frac{1}{2}\rho v^2 . \pi R^2 - \frac{1}{2}\rho v^2 . 2\pi \left[\int_0^a \frac{r^2}{4h^2} . r\,dr + \int_a^R \frac{a^4}{4h^2}\frac{1}{r^2} r\,dr \right]$$

$$= \frac{1}{2}\rho v^2 . \pi R^2 - \frac{1}{2}\rho v^2 . 2\pi \left[\frac{a^4}{16h^2} + \frac{a^4}{4h^2} \ln \frac{R}{a} \right]$$

$$= \frac{1}{2}\pi\rho v^2 \left[R^2 - \frac{a^4}{8h^2} \right] - \frac{1}{2}\rho v^2 . 2\pi \frac{a^4}{4h^2} . \ln \frac{R}{a}$$

$$= \frac{1}{2}\pi\rho v^2 \left[R^2 - \frac{a^4}{8h^2} \right] + \rho v^2 \pi \frac{a^4}{4h^2} \ln \frac{a}{R}$$

This force can be negative when ~~$\frac{a^2}{h} << R$~~

$\frac{a^2}{h} >> R$ & $a << R$.

Using $a = 0.01m$, $R = 0.05m$, $h = 10^{-3}m$, $m = 0.01\,kg$, we get velocity as:

$V = 1.73$ m/s.

d) Using the control volume shown in Fig 1(b), ~~we~~ we apply the Reynold's transport theorem. Here the parameter 'h' is ~~not consta~~ a function of time.

$$\frac{d}{dt}\int_{V_t} \rho\, dV = \cancel{\int_V \frac{d\rho}{dt} dV}^{\,0} + \int \rho \bar{V}.\hat{n}\, ds$$

$$\Rightarrow \frac{d}{dt}\left(\rho \pi r^2 h\right) = -\rho V . \pi a^2 + \rho V_r . 2\pi r h$$

$\Rightarrow$

$$\Rightarrow \rho\pi r^2 \frac{dh}{dt} = \rho\pi a^2 v - 2\pi r h \rho v_r$$

$$\Rightarrow v_r = \frac{-r\dot{h}}{2h} + \frac{a^2 v}{2hr}, \quad r > a$$

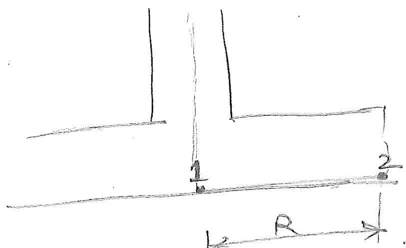
also, considering $v_r = c_1(\dot{h}, h)\, r \quad r < a$, we get

$$c_1(\dot{h}, h)\, r \Big|_{r=a} = -\frac{a\dot{h}}{2h} + \frac{av}{2h}$$

$$\Rightarrow c_1 = \frac{v}{2h} - \frac{\dot{h}}{2h}$$

$$\therefore v_r = \begin{cases} \left(\dfrac{v - \dot{h}}{2h}\right) r & r < a \\[2ex] \dfrac{-r\dot{h}}{2h} + \dfrac{a^2 v}{2hr} & r \geq a \end{cases}$$

Applying Bernoulli's theorem over streamline ~~in Fig 2~~,



$$\int_1^2 \frac{dv}{dt}\, ds + \left[\frac{v^2}{2} + \frac{p}{\rho}\right]_1^2 = 0$$

$$\Rightarrow \int_1^2 \frac{dv}{dt}\, ds = \int_0^a -\frac{\ddot{h}}{2h} r\, dr + \int_a^R \left[-\frac{r\ddot{h}}{2h} - \frac{\dot{h}}{h^2}\left(\frac{a^2 v - r^2 \dot{h}}{2r}\right)\right] r\, dr$$

$$= p_1 + \cancelto{0}{\frac{v_1^2}{2}} - p_2 - \frac{v_2^2}{2}$$

$$= p_1 - p_2 - \left(\frac{a^2 v - r^2 \dot{h}}{2h}\right)^2$$

2.) a) Boundary conditions

~~$\left.\bar{V}\right|_{x\to-\infty} = V_\infty \hat{i} + 0\hat{j}$~~

$$\left.\vec{V}\right|_{x\to-\infty} = U\hat{i} + 0\hat{j}$$

& $$\left.\bar{V}\cdot\hat{n}\right|_{r=R} = 0$$

$$\Rightarrow \left.\frac{\partial\psi}{\partial y}\right|_{x\to-\infty} = U$$

& $$\left.\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right|_{r=R} = 0$$

b) $$\psi(r,\theta) = r^{\alpha}\sin\theta.$$

$$\nabla^2\psi = 0$$

$$\Rightarrow \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$$

$$\Rightarrow \alpha(\alpha-1)r^{\alpha-2}\sin\theta + \alpha r^{\alpha-2}\sin\theta - r^{\alpha-2}\sin\theta = 0$$

$\because$ this must be true for any r & θ,

$$\Rightarrow \alpha(\alpha-1) + \alpha - 1 = 0$$

$$\Rightarrow \alpha = 1, -1$$

$$\therefore \psi = Ar\sin\theta + \frac{B}{r}\sin\theta = Ay + \frac{By}{r^2} = Ay + \frac{By}{x^2+y^2}$$

Applying boundary conditions,

$$\Rightarrow \left.\frac{\partial\psi}{\partial y}\right|_{x\to-\infty} = A = U$$

& $\frac{1}{r}\frac{\partial\psi}{\partial\theta}\Big|_R = \left[A\cos\theta + \frac{B}{r^2}\cos\theta\right]_{r=R} = 0$

$\Rightarrow A + \frac{B}{R^2} = 0$

$\Rightarrow B = -AR^2 = -UR^2$

$\therefore \psi = Ur\sin\theta - U\frac{R^2}{r}\sin\theta$

$= UR\sin\theta\left(\frac{r}{R} - \frac{R}{r}\right)$

$V_\theta = \frac{-\partial\psi}{\partial r}$

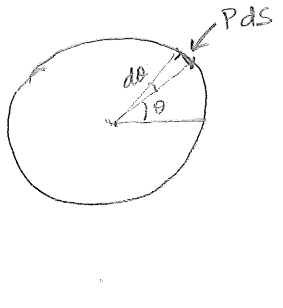
$= -UR\sin\theta\left(\frac{1}{R} + \frac{R}{r^2}\right) = U\sin\theta\left(1 + \frac{R^2}{r^2}\right)$

Computing pressure on cylinder using Bernoulli's theorem,

$P_\infty + \frac{1}{2}\rho U^2 = P(\theta) + \frac{1}{2}\rho U^2\sin^2\theta\left(1 + \frac{R^2}{R^2}\right)$

$\Rightarrow \cancel{P_\theta - P_\infty}\ P(\theta) = P_\infty + \frac{1}{2}\rho U^2\left[1 - 2\sin^2\theta\right]$

The net force on cylinder :



$dF_x = -PdS\cos\theta$, $dF_y = -PdS\sin\theta$

where $ds = rd\theta$, $P = P(\theta)$

$F_x = \int dF_x = \int_0^{2\pi}\left[P_\infty + \frac{1}{2}\rho U^2(1 - 2\sin^2\theta)\right]\cos\theta\, d\theta$

$= \int_0^{2\pi}\left(P_\infty + \frac{1}{2}\rho U^2\right)\cos\theta\, d\theta - \frac{1}{2}\rho U^2\int_0^{2\pi} 2\sin^2\theta\cos\theta\, d\theta$

$= 0$

$|||^{ly}$, $F_y = \int dF_y = 0$. There is no net force because the flow is inviscid. Hence, the drag force is zero.