AFM HOMEWORK 3

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1)

	$\Delta P/L$	ρ	$\bar{v}_0$	R	R_1	μ_1	μ_2	σ
M	1	1	0	0	0	1	1	1
L	-2	-3	1	1	1	-1	-1	0
T	-2	0	-1	0	0	-1	-1	-2

Rank r = 3

n = 8

n − r = 5 no. of π terms

$$\pi_1 = \frac{\Delta P}{L}\rho^a \bar{v}_0^b R_1^c$$

$$M^0L^0T^0 = ML^{-2}T^{-2} \times M^aL^{-3a}T^0 \times L^bT^{-b} \times L^c$$

$$= M^{1+a} L^{-2-3a+b+c} T^{-2-b}$$

$\therefore a = -1,\ b = -2$; $\cancel{-2+3-2}$ $-2-3a+b+c = 0$

$\Rightarrow -2+3-2+c = 0$

$\Rightarrow c = 1$

$$\pi_1 = \frac{\Delta P/L \cdot R_1}{\rho \bar{v}_0^2} = \frac{\Delta P R_1}{L\rho \bar{v}_0^2}$$

$$\pi_2 = \mu_1 \rho^a \bar{v}_0^b R_1^c$$

$$M^0L^0T^0 = ML^{-1}T^{-1} \times M^aL^{-3a} \times L^bT^{-b} \times L^c$$

$$= M^{1+a} L^{-1-3a+b+c} T^{-1-b}$$

$\therefore a = -1,\ b = -1,\ b+c-3a-1 = 0 \Rightarrow -1+c+3-1 = 0$

$\Rightarrow c = -1$

$$\pi_2 = \frac{\mu_1}{\rho \bar{v}_0 R_1}$$

Similarly $\pi_3 = \dfrac{\mu_2}{\rho \bar{v}_0 R_1}$

$\pi_4 = \sigma \rho^a \bar{v}_0^b R_1^c \Rightarrow M^0L^0T^0 = MT^{-2} \times M^aL^{-3a} \times L^bT^{-b} \times L^c$

$= M^{1+a} \cdot L^{b-3a+c} T^{-2-b}$

$\therefore b = -2,\ a = -1;\ b-3a+c = 0 \Rightarrow c = -1$ $\quad\therefore \pi_4 = \dfrac{\sigma}{\rho \bar{v}_0^2 R_1}$

$\pi_5 = R \rho^a \vartheta_0^b R_1^c$

$= L \times M^a L^{-3a} \times L^b T^{-b} \times L^c$

$M^0 L T^0 = M^a L^{1-3a+b+c} T^{-b}$

$a = 0,\ b = 0\ ;\ 1 - 3a + b + c = 0$

$\Rightarrow c = -1$

$\therefore \pi_5 = \frac{R}{R_1}$

$\therefore \pi_1 = \frac{\Delta P R_1}{L \rho \vartheta_0^2},\ \pi_2 = \frac{\mu_1}{\rho \vartheta_0 R_1},\ \pi_3 = \frac{\mu_2}{\rho \vartheta_0 R_1},\ \pi_4 = \frac{\sigma}{\rho \vartheta_0^2 R_1}$

$\pi_5 = \frac{R}{R_1}$

b) π_2 & π_3 correspond to Reynolds no. π_4 correspond to Weber no.

Among all these the Weber no. is important whether waves will develop.

Interfacial tension is the work force which must be expanded to increase size of the interface between 2 adjascent phases which do not mix completely with one another. That means if no external forces apply, the liquid phases minimize the size of their ~~surf~~ interface.

Therefore waves will not form unless ~~inter~~ inertia forces are very small compared to surface tension, which means if the Weber no. is very small.

Weber no. << 1

$\Rightarrow \pi_4 >> 1$

$\Rightarrow \frac{\sigma}{R} >> \rho \vartheta_0^2$

c) Since the fluids have same density $\rho_1 = \rho_2 = \rho$, gravity effects can be neglected and it is reasonable to assume that gravity will not contribute to the formation of waves at interface

d) $\vec{v} = (0, 0, v_z(r))$

Navier Stokes eq

$$-\frac{\partial p}{\partial r} = 0$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} = 0$$

$$-\frac{\partial p}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = 0$$

where $\mu = \begin{cases} \mu_1 & \text{if } 0 < r < R_1 \\ \mu_2 & \text{if } R_1 < r < R \end{cases}$

Assuming the interface is cylindrical, the boundary condition for at the interface

$$v_z\big|_{r=R} = 0$$

For no slip at interface $\tau_{rz}\big|_{r=R_1^-} = \tau_{rz}\big|_{r=R_1^+}$

$$\Rightarrow \mu_1 \frac{\partial v_z}{\partial r}\bigg|_{r=R_1^-} = \mu_2 \frac{\partial v_z}{\partial r}\bigg|_{r=R_1^+}$$

also $\frac{\partial v_z}{\partial r}\Big|_{r=0} = 0$ from geometry

$p(z) = p_2(z) + \frac{c}{R_1}$

$\frac{\partial p_1}{\partial z} = \frac{\partial p_2}{\partial z} = \frac{\Delta p}{L}$

e) Considering the z component of Navier Stokes eq. we know the solutions are of form: $\frac{\partial p}{\partial z} = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{\Delta p}{L} r = \frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$

Integrating, $v_z = \frac{1}{4\mu_1}\frac{\Delta p}{L} r^2 + a_1 \ln(r) + b_1$ if $0 < r < R_1$

$v_z = \frac{1}{4\mu_2}\frac{\Delta p}{L} r^2 + a_2 \ln(r) + b_2$ if $R_1 < r < R$

Using BC

$\frac{\partial v_z}{\partial r}\Big|_{r=0} = 0$ $\Rightarrow$ gives $a_1 = 0$

$\mu_1 \frac{\partial v_z}{\partial r}\Big|_{r=R_1^-} = \mu_2 \frac{\partial v_z}{\partial r}\Big|_{r=R_1^+}$ gives $a_2 = 0$

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$v_2|_{r=R} = 0$ gives $b_2 = \frac{1}{4\mu_2}\frac{\Delta p}{L} R^2$

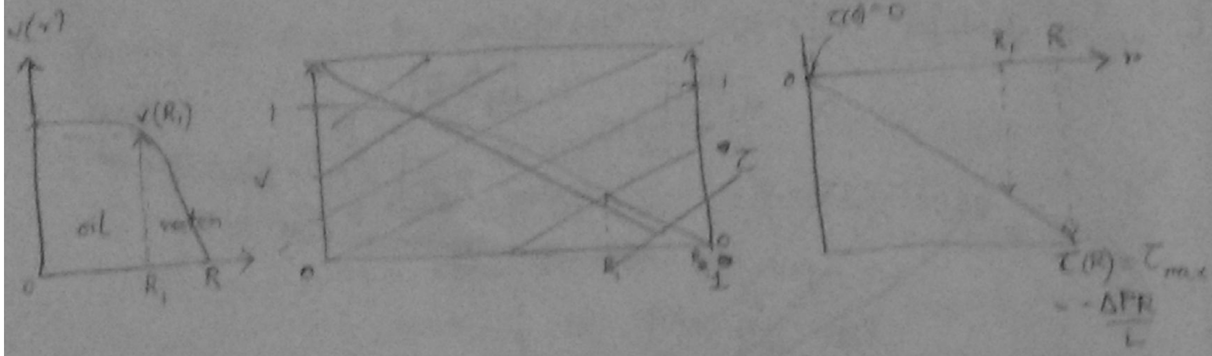
$v_1|_{r=R_1} = v_2|_{r=R_1}$ gives $b_1 = -\frac{1}{4\mu_1}\frac{\Delta p}{L}(R_1^2 - r^2)\ \frac{1}{4\mu_2}\frac{\Delta p}{L}(R^2 - R_1^2)$

$\therefore\ v_2 = \frac{1}{4\mu_2}\frac{\Delta p}{L}(R^2 - r^2)$ if $R_1 < r < R$

$v_1 = \frac{1}{4\mu_1}\frac{\Delta P}{L}(R_1^2 - r^2) + \frac{1}{4\mu_2}\frac{\Delta p}{L}(R^2 - R_1^2)$ if $0 < r < R_1$

$v_2|_{\text{interface}} = \frac{1}{4\mu_2}\frac{\Delta p}{L}(R^2 - R_1^2)$

The shear stress $\tau_{rz} = \mu\frac{\partial v_z}{\partial r} = -\frac{1}{2}\frac{\Delta p}{L} r$



f) Volume flow rate $Q_o = \int_0^{R_1} 2\pi r v_z\, dr = \frac{2\pi}{4}\left(\frac{\Delta P}{L}\right)\int_0^{R_1} \frac{r}{\mu_1}(R_1^2 - r^2) + \frac{r}{\mu_2}(R^2 - R_1^2)\, dr$

$\cancel{\frac{\pi}{4}\left(\frac{\Delta P}{L}\right)} R_1 = \frac{\pi}{8\mu_1}\frac{\Delta P}{L} R_1^4 + \frac{\pi}{4\mu_2}\frac{\Delta P}{L}(R^2 - R_1^2)R_1^2$

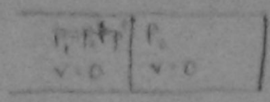
$Q_w = \int_{R_1}^{R} 2\pi r v_z\, dr = \int_{R_1}^{R} \frac{2\pi}{4\mu_2}\frac{\Delta P}{L}(R^2 - r^2)\cdot r\, dr$

$= \frac{\pi}{2\mu_2}\frac{\Delta P}{L}\int_{R_1}^{R}(R^2 r - r^3)\, dr = \frac{\pi}{2\mu_2}\frac{\Delta P}{L}\left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_{R_1}^{R}$

$= \frac{\pi}{2\mu_2}\frac{\Delta P}{L}\left[\left(\frac{R^4}{4}\right) - \left(\frac{R^2 R_1^2}{2} - \frac{R_1^4}{4}\right)\right]$

$= \frac{\pi}{8\mu_2}\frac{\Delta P}{L}\left[R^4 - 2R_1^2 R^2 + R_1^4\right] = \frac{\pi}{8\mu_2}\frac{\Delta P}{L}\left(R^2 - R_1^2\right)^2$

2)



$p_1 = p_0 + p'$

$p_0 >> p'$

Given $(p_1 - p_0)/p_0 \ll 1$

①

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

$$\frac{\partial}{\partial t}(\rho \vec{v}) + \nabla \cdot (\rho \vec{v} \otimes \vec{v} + p\mathbf{I}) = \rho \vec{b}$$

$$\frac{\partial}{\partial t}(\rho E) + \nabla \cdot ((\rho E + p)\vec{v}) = \vec{v} \cdot \rho \vec{b}$$

$\rho = \rho_0 + \rho'$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho_0 u) + \frac{\partial}{\partial x}(\rho' u)$$ $= 0$ neglecting since $\rho' \ll \rho_0$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0 \quad \text{--- (1)}$$

$P = P_0 + P'$

Neglecting higher order term $v \otimes v$, and no body forces

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial P}{\partial x} = 0 \quad \text{--- (2)}$$

Let $\rho_n = \dfrac{\rho - \rho_0}{\rho_0} \Rightarrow \rho = \rho_0(1 + \rho_n)$

$$\therefore \frac{\partial \rho}{\partial t} = \rho_0 \frac{\partial \rho_n}{\partial t}$$

Eq 1 becomes $\dfrac{\partial \rho_n}{\partial t} + \dfrac{\partial u}{\partial x} = 0$ --- (3)

For ideal gas $\dfrac{P}{\rho^\gamma} = \dfrac{P_0}{\rho_0^\gamma}$ where $\gamma = \dfrac{c_p}{c_v}$

$$\Rightarrow \frac{P}{(1+\rho_n)^\gamma \rho_0^\gamma} = \frac{P_0}{\rho_0^\gamma} \Rightarrow P = P_0(1+\rho_n)^\gamma$$

$$P \approx P_0(1 + \gamma \rho_n)$$

$$\Rightarrow \frac{\partial P}{\partial x} = \gamma P_0 \frac{\partial \rho_n}{\partial x}$$

$\therefore$ Eq (2) becomes

$$\rho_0 \frac{\partial u}{\partial t} + \gamma P_0 \frac{\partial \rho_n}{\partial x} = 0$$

$$\Rightarrow \frac{\partial u}{\partial t} + \frac{\gamma P_0}{\rho_0} \frac{\partial \rho_n}{\partial x} = 0$$

$$\Rightarrow \frac{\partial u}{\partial t} + c^2 \frac{\partial \rho_n}{\partial x} = 0 \quad \text{where } c^2 = \frac{\gamma P_0}{\rho_0}$$

$\therefore$ System of equations to solve: we replace ρ_n by ρ for simplification

$$\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} = 0 \quad \text{--- (4)}$$

$$\frac{\partial u}{\partial t} + c^2 \frac{\partial \rho}{\partial x} = 0 \quad \text{--- (5)}$$

$(4) + ((5)/c)$ gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} + \frac{1}{c}\frac{\partial u}{\partial t} + c\frac{\partial \rho}{\partial x} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{u}{c} + \rho\right) + c\frac{\partial}{\partial x}\left(\frac{u}{c} + \rho\right) = 0$$

Let $J = \frac{u}{c} + \rho$

$$\frac{\partial J}{\partial t} + c\frac{\partial J}{\partial x} = 0$$

$(4) - ((5)/c)$ gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} - c\frac{\partial \rho}{\partial x} - \frac{1}{c}\frac{\partial u}{\partial t} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\rho - \frac{u}{c}\right) + c\frac{\partial}{\partial x}\left(\rho - \frac{u}{c}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{u}{c} - \rho\right) - c\frac{\partial}{\partial x}\left(\frac{u}{c} - \rho\right) = 0$$

Let $K = \frac{u}{c} - \rho$

$$\frac{\partial K}{\partial t} - c\frac{\partial K}{\partial x} = 0$$

There the new system to be solved is

$$\frac{\partial J}{\partial t} + c\frac{\partial J}{\partial x} = 0$$

$$\frac{\partial K}{\partial t} - c\frac{\partial K}{\partial x} = 0$$

$J = \frac{u}{c} + \rho_n$

$\rho_n = \frac{\rho - \rho_0}{\rho_0}$

Using characteristic line method for eq (3)

$$\frac{\partial J}{\partial t} + c\frac{\partial J}{\partial x} = 0$$

$$\frac{dJ}{ds} = \frac{\partial J}{\partial x}\frac{dx}{ds} + \frac{\partial J}{\partial t}\frac{dt}{ds}$$

$$\frac{dx}{ds} = c, \quad \frac{dt}{ds} = 1$$

if $\frac{dx}{ds} = c$ & $\frac{dt}{ds} = 1$

we get

$$\frac{dJ}{ds} = \frac{\partial J}{\partial t} + c\frac{\partial J}{\partial x} = 0$$

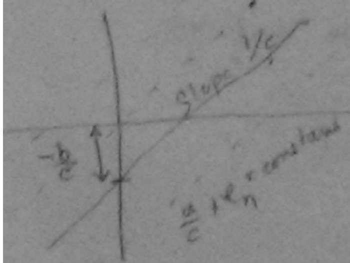
Assume $t(0) = 0$

$t = s$

$x = ct + b$

~~$x = ct + b$~~

$\Rightarrow \frac{x}{c} - \frac{b}{c} = t$



Slope $= \frac{1}{c}$ (+ve)

Depending on value of b (I.C.) we will have family of lines shown by dotted lines

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0$$

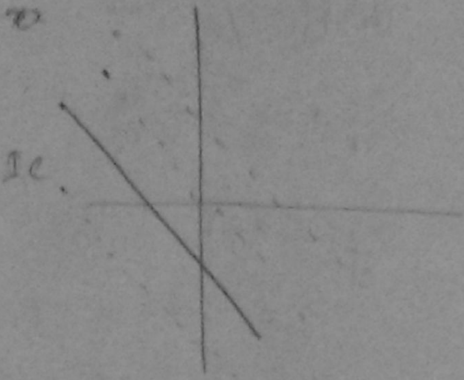
$$\frac{dx}{ds} = -c;\ \frac{dt}{ds} = 1$$

$x = -ct + d$

$-\frac{x}{c} - \frac{d}{c} = t$

Slope $= -\frac{1}{c}$

We get dotted lines base on IC.



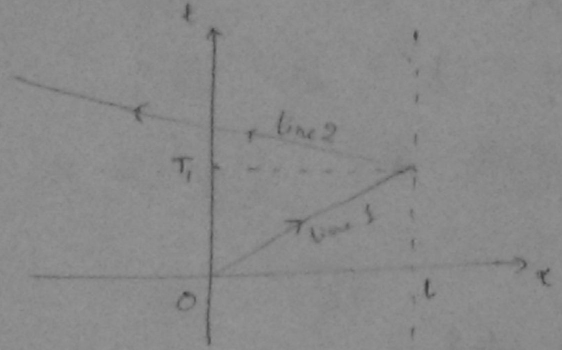
our problem is a combination of these two graphs

for time = 0 to T_1 we get forward curve

for time = T_1 to ∞ we get backward curve

At $t=0, x=0$ (I.C)

At $t=T_1$, $x=x_1$ (wall)



line 1 eqn

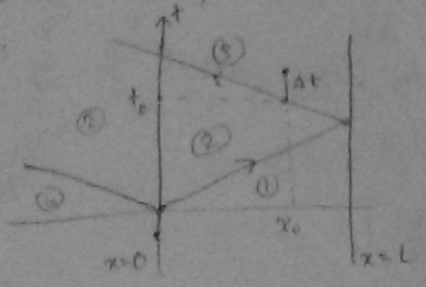
$t = \frac{x}{c}$

line 2 eqn

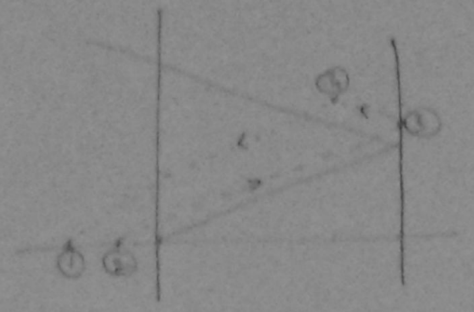
$x = -c(t-T_1) + L$

We want to know solution after the wave has passed

Therefore we start at a point and measure u, f after the wave has passed.



We are at $x=x_0$, then we wait for time t_0 so that wave passes and then wait for time Δt (very small) so that we are behind wave and then measure u & f



(1) – (4) $\frac{u}{c} + \rho_n = \text{const}$.

(3) – (4) $\frac{u}{c} - \rho_n = \text{const}$.

(2) – (3) $\frac{u}{c} + \rho_n = \text{const}$.

at (1) $\rho = \rho_1$

$u_1 = 0$

at (2) $\rho = \rho_1$

$u_2 = 0$

$$\frac{u_1}{c} + \rho_1 = \frac{u_4}{c} + \rho_4$$

$$\boxed{\rho_1 = \frac{u_4}{c} + \rho_4} \quad \text{— (6)}$$

$$\frac{u_2}{c} + \rho_2 = \frac{u_3}{c} + \rho_3$$

$$\rho_2 = \frac{u_3}{c} + \rho_3$$

$u_3 = 0$ at boundary

$$\rho_3 = \rho_2 = \rho_1$$

between (3) & (4)

$$\frac{u_3}{c} - \rho_3 = \frac{u_4}{c} - \rho_4$$

$$-\rho_1 = \frac{u_4}{c} - \rho_4 \quad \text{— (7)}$$

Eq (6) & (7) have 2 unknowns

$$\rho_1 = \frac{u_4}{c} + \rho_4$$

$$-\rho_1 = \frac{u_4}{c} - \rho_4$$

Adding we get

$u_4 = 0$

~~Substract~~. Eq. (6) – (7)

$$2\rho_1 = 2\rho_4 \Rightarrow \rho_1 = \rho_4$$

To find pressure

$$\frac{P_4}{\rho_4^\gamma} = \frac{P_1}{\rho_1^\gamma} \Rightarrow P_1 = P_4$$

∴ Just behind wave,

$u = 0$, $P = P_1$