ADVANCED FLUID MECHANICS

Homework 4: Navier Stokes Equation & Boundary Layer

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1. **Fluid stream impinging a plate**

   a) The stream potential can also be expressed in Cartesian coordinates as:

$$\psi = U r^2 \sin(2\theta) = Ur^2 \cdot 2\sin\theta\cos\theta = 2\mathrm{U}(x^2+y^2)\frac{x\,y}{x^2+y^2} = 2\,U\,xy$$

The velocity in the Cartesian coordinates can be calculated as:

$$v_x = \frac{\partial\psi}{\partial y} = 2\mathrm{Ux} \text{ and } v_y = -\frac{\partial\psi}{\partial x} = -2\mathrm{Uy}$$

At $y=0$, we have:

$$v_x = 2\mathrm{Ux} \text{ and } v_y = 0$$

Hence, it satisfies the boundary condition for an inviscid flow over the plane boundary.
The Stream function can be checked for irrotationality as:

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} = 0$$

Hence, the flow is irrotational. The Bernoulli's equation can be applied across the streamlines.
The velocity at the origin is:

$$v_x = 0 \text{ and } v_y = 0$$

which is a stagnation point. Assuming the pressure at the origin, the stagnation pressure as the reference pressure p_0 , and in the absence of any body forces, we can apply Bernoulli equation as:

$$p_0 = p_{xy}+\frac{1}{2}\rho v^2$$

$$\Rightarrow p_{xy} = p_0-\frac{1}{2}\rho v^2$$

$$\Rightarrow p_{xy} = p_0-\frac{1}{2}\rho\cdot 4\,U^2(x^2+y^2)$$

$$\Rightarrow p_{xy} = p_0-2\,\rho\cdot U^2 r^2$$

   b) The Navier Stokes equation for a steady flow can be written as follows:

$$\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla p+\mu\nabla^2\boldsymbol{v}$$

Evaluating individual terms:

$$\boldsymbol{v}\cdot\nabla\boldsymbol{v} = \left[v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}\right]\boldsymbol{v} = \begin{bmatrix}4U^2x\\4U^2y\end{bmatrix} ,$$

$$-\nabla p = = \begin{bmatrix} -4\rho U^2 x \\ -4\rho U^2 y \end{bmatrix} \text{, and } \nabla^2 \mathbf{v} = \mathbf{0}$$

The above terms satisfy the Navier-Stokes equation on substitution. Note that the viscous term is zero, and the pressure gradient completely balances the acceleration term.

c) The modified velocity field with $u=2Uxf'(y)$ will satisfy the continuity equation under the condition:

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y} = 0$$

$$\Rightarrow 2Uf'(y)+\frac{\partial v_y}{\partial y} = 0$$

$$\Rightarrow v_y = -2Uf(y)+g(x)$$

Applying boundary conditions for no-slip condition over the wall, we get:

$$v_y(y=0) = v_x(y=0) = 0$$

$$\Rightarrow -2\mathrm{U} f(0)+g(x)=0 \ \ \& \ \ 2Uxf'(0)=0$$

$$\Rightarrow g(x)=k \ \ \& \ \ f'(y)=0 \text{ , where } k \text{ is a constant.}$$

And the constant can be absorbed by the function $f(y)$, so $v_y = -2Uf(y)$

Hence, the boundary conditions are $f(0)=0$ & $f'(0)=0$

d) The y-momentum equation is:

$$\rho\left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}\right)v_y = -\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)v_y$$

On substituting the new velocity field, we obtain:

$$\rho\left(2Uf'(y)\frac{\partial}{\partial x}-2Uf(y)\frac{\partial}{\partial y}\right)[-2Uf(y)] = -\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)[-2Uf(y)]$$

$$\Rightarrow 4\rho U^2 f(y)f'(y) = -\frac{\partial p}{\partial y}-2\mu Uf''(y)$$

$$\Rightarrow \frac{\partial p}{\partial y} = -2\mu Uf''(y)-4\rho U^2 f(y)f'(y)$$

On integrating the above obtained expression for pressure, we have:

$$\int\frac{\partial p}{\partial y}dy = \int\left[-2\mu Uf''(y)-4\rho U^2 f(y)f'(y)\right]dy$$

$$\Rightarrow p_{xy} = \left[-2\mu Uf'(y)-2\rho U^2 f^2(y)\right]+h(x)$$

The pressure gradient in both velocity potentials must match at large y . Hence, at large y , we have:

$$\lim_{y\to\infty} p_{xy}=p_0-2\rho U^2(x^2+y^2)$$

$$\Rightarrow \lim_{y\to\infty} f^2(y)=y^2 \text{ , and } \lim_{y\to\infty}[h(x)-2\mu Uf'(y)] = -2\rho U^2x^2+p_0$$

$$\Rightarrow \lim_{y\to\infty} f(y)=y \text{ , and } h(x)-2\mu U\left[\lim_{y\to\infty} f'(y)\right] = -2\rho U^2x^2+p_0$$

Since, $f(y)$ approaches y assymptotically, we must have $f'(y)$ approach 1 at ∞ .

Hence, the function $h(x)$ is:

$$h(x) = p_0 + 2\mu U - 2\rho U^2 x^2$$

and the pressure field is:

$$p_{xy} = p_0 - 2\mu U[1 - f'(y)] - 2\rho U^2 [f^2(y) + x^2]$$

e) The x-momentum equation is:

$$\rho\left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}\right) v_x = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v_x$$

$$\Rightarrow \rho\left[(2Uxf'(y))\frac{\partial}{\partial x} + (-2Uf(y))\frac{\partial}{\partial y}\right](2Uxf'(y))$$

$$= 4\rho U^2 x + \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(2Uxf'(y))$$

$$\Rightarrow 4\rho U^2 x\left[f'^2(y) - f(y)f''(y)\right] = 4\rho U^2 x + 2\mu U x f'''(y)$$

$$\Rightarrow f''' + \frac{2\rho U}{\mu}\left(f \cdot f'' - f'^2 + 1\right) = 0$$

Since, this a 3rd degree ODE, and we have three boundary conditions, the system has a unique solution.

2. Kármán-Pohlhausen approximation to Boundary Layer

The given velocity profile is:

$$\frac{u}{U} = a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^2$$

and the boundary conditions :

$$u = 0 \quad \text{at} \quad y = 0$$

$$u = U \;,\; \frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = \delta$$

On substituting the given velocity profile into the given boundary conditions, we get:

$$\frac{u}{U} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$$

Evaluate momentum thickness of the boundary layer using:

$$\frac{\theta}{\delta} = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right)$$

$$\Rightarrow \theta = \frac{2}{15}\delta$$

Also, the shear stress at $y = 0$ can be evaluated as:

$$\frac{\tau_0}{\rho} = \frac{2\nu U}{\delta}$$

$$\frac{d}{dx}U^2\theta = \frac{\tau_0}{\rho}$$

since U does not vary with x ,

$$\Rightarrow U^2 \frac{d}{dx}\left(\frac{2}{15}\delta\right) = \frac{2\nu U}{\delta}$$

$$\Rightarrow \delta^2 = 30\frac{\nu x}{U}$$

$$\Rightarrow \frac{\delta}{x} = \frac{5.4772}{\sqrt{Re_x}}$$

We have the solution for boundary layer thickness and hence, the velocity profile in the boundary layer.

On comparison with Blassius solution and General momentum solution with a 3rd degree polynomic velocity profile, we can see that the δ/x is still proportional to $Re^{-1/2}$. But the constant of relation changes changes. For Blassius solution and with cubic polynomial, the boundary layer thickness is:

$$\frac{\delta}{x} = \frac{5}{\sqrt{Re_x}} \quad \text{(Blassius solution)}$$

$$\frac{\delta}{x} = \frac{4.64}{\sqrt{Re_x}} \quad \text{(cubic profile)}$$

Hence, $\delta_{cubic} < \delta_{Blassius} < \delta_{quadratic}$

The shear stress on the plate for all three cases are ass follows:

$$\frac{\tau_0}{\rho} = \frac{0.7303}{\sqrt{Re_x}} \quad \text{(quadratic profile)}$$

$$\frac{\tau_0}{\rho} = \frac{0.6466}{\sqrt{Re_x}} \quad \text{(cubic profile)}$$

$$\frac{\tau_0}{1/2\rho U^2} = \frac{0.664}{\sqrt{Re_x}} \quad \text{(Blassius solution).}$$

$$\Rightarrow \tau_{cubic} < \tau_{Blassius} < \tau_{quadratic}$$

Hence, it can concluded that the solution obtained using the quadratic boundary layer is thicker and creates more drag.

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