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MSc Computational Mechanics

A.F.M. → H.W.-4

answer - 1 →.

(a) $\psi(r, \theta) = U r^2 \sin 2\theta$

$r = x^2 + y^2$

$\sin 2\theta = 2 \sin\theta \cos\theta$

$= \frac{2y}{\sqrt{x^2+y^2}} \frac{x}{\sqrt{x^2+y^2}}$ (By Pythagorus theorem)

$= \frac{2xy}{x^2+y^2}$

$\psi(x, y) = \frac{U(x^2+y^2)2xy}{(x^2+y^2)} = 2xyU$

$\psi(x, y) = 2xyU$

$u = \frac{\partial\psi}{\partial y} = 2xU$; $v = -\frac{\partial\psi}{\partial x} = -2yU$

Boundry Conditions,

① at $x = 0$ (along Y axis) $u = 0$ (symmetry + flow velocity)

$u\big|_{x=0} = 2 \times 0 \times U = 0$ (satisfies)

② at $y = 0$ (along X axis) $v = 0$ (cannot penetrate the boundry)

$v\big|_{y=0} = -2 \times 0 \times U = 0$ (satisfied)

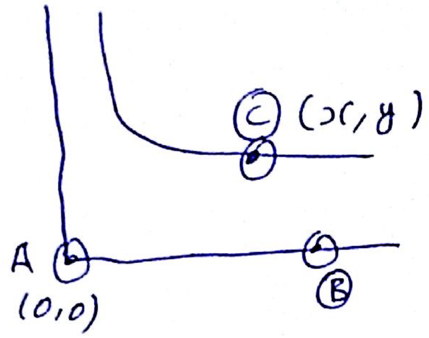
This is assumed as Ideal fluid, so it can slip on boundaries, ∴ no slip boundry cannot be applied

$u = 2xU$

$v = -2yU$

$\nabla \times \bar{v} = 0$ irrotational flow, no, bernoullis theorem can be applied between any two points in space.

between A & C



$$P_0 + 0 = P_x + \frac{1}{2}\rho |\bar{V}|^2$$

$$P_x = P_0 - \frac{1}{2}\rho\left[(2xU)^2 + (-2yU)^2\right]$$

$$\boxed{P = P_0 - 2\rho U^2(x^2+y^2)}$$

(b) Navier - Stokes for 2D flows and no body forces

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad - (1)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad - (2)$$

eqn (1)

L.H.S.	R.H.S.
$\rho(0 + (2xU)(2U) + 0)$	$-\frac{\partial P}{\partial x} + \mu(0+0)$
$4U^2x\rho$	$2\rho U^2(2x)$
$4U^2x\rho$	$4\rho U^2 x$

L.H.S = R.H.S.

eq'n (2)

L.H.S.	R.H.S.
$\rho[0 + 0 + (-2yU)(-2U)]$	$-\frac{\partial P}{\partial y} + \mu(0+0)$
$+4yU^2\rho$	$(2)\rho U^2(2y)$
$4yU^2\rho$	$4\rho U^2 y$

L.H.S. = R.H.S.

Hence, former velocity and pressure satisfy the navier-strokes eq'n's

B.C. for viscous problem

no-slip on $(y=0)$

So $u|_{y=0} = 0$

But $u|_{y=0} = 2xU \neq 0$ so this condition is not satisfied

(c) $u = 2xUf'$ f is a function of only y

continuity eq^n

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial y} = -2Uf'$$

$$\boxed{v = -2Uf}$$

Based on Boundry conditions

① No normal velocity at boundry (stationary)

$v|_{y=0} = 0 \Rightarrow f(0) = 0$

② No slip at boundry

$u|_{y=0} = 0 \Rightarrow f'(0) = 0$

③ at large y

$u|_{y\to\infty} = 2xU \Rightarrow f'(\infty) = 1$

$v|_{y\to\infty} = -2uf \Rightarrow f(\infty) = y$

(d) Y-momentum with $u = 2xUf'$ $v = -2Uf$

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\rho(2xUf')(0) + \rho(-2Uf)(-2Uf') = -\frac{\partial P}{\partial y} + \mu(0 + (-2Uf''))$$

$$4U^2 ff'\rho = -\frac{\partial P}{\partial y} - 2U\mu f''$$

R.H.S. = only function of y so integrating directly

$$P = -2U^2\rho f^2 - 2U\mu f' + m(x)$$

$$P(x,y) = -2U^2\rho f^2 - 2U\mu f' + m(x)$$

recovery and old flow at $y \to \infty$

So P should approach old pressure value.

$$P_0 - 2\rho U^2(x^2+y^2) = -2U^2\rho y^2 - 2U\mu + m(x)$$

$(\because F(\infty) = y$ & $F'(\infty) = 1)$

$$\therefore m(x) = P_0 - 2\rho U^2 x^2 + 2U\mu$$

$$\boxed{\therefore P(x,y) = P_0 - 2\rho U^2 f^2 + 2\mu U(1-f') - 2\rho U^2 x^2}$$

(e) x- momentum

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu(0 + 2Uxf''')$$

$$\rho(2xUf')(2Uf') - \rho(2Uf)(2Uxf'') = -\frac{\partial P}{\partial x} + \mu 2Uxf'''$$

$$\rho 4\cancel{x}U^2(f')^2 - 4U^2\cancel{x}\rho ff'' = -2\rho U^2(2\cancel{x}) + 2\mu U\cancel{x}f'''$$

$(\div$ by $4\rho U^2)$

$$(f')^2 - ff'' = 1 + \frac{\nu}{2U}f''' \qquad \left(\nu = \frac{\mu}{\rho}\right)$$

$$\boxed{\frac{\nu}{2U}f''' + ff'' - (f')^2 + 1 = 0}$$

B.C.'s from (c) part.

① $f(\infty) = y$

② $f'(0) = 0$

③ $f(0) = 0$

So this is a third order ODE in f with 3 boundry conditions but it is non-linear. This can be solved numerically with R-K method or any other method.

Solution 2:-

$$\frac{u}{U} = a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^2$$

B.C. given

1. $u = 0$ at $y = 0$

$$\Rightarrow 0 = a + 0 + 0$$

$$\boxed{a = 0}$$

2. $u = U$ at $y = \delta$

$$\Rightarrow 1 = 0 + b + c$$

$$\boxed{b + c = 1}$$

3. $\frac{\partial u}{\partial y} = 0$ at $y = \delta$

$$\frac{1}{U}\frac{du}{dy} = \frac{b}{\delta} + \frac{c}{\delta^2}2y$$

at $y = \delta$

$$0 = \frac{b}{\delta} + \frac{2c}{\delta}$$

$$\boxed{b + 2c = 0}$$

$$\Rightarrow \boxed{b = 2 \quad ; \quad c = -1}$$

$$\boxed{u = U\frac{2y}{\delta} - \frac{Uy^2}{\delta^2}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

for flat plate, these eq^n's when integrated and combined.

$$\frac{d}{dx}\int_0^\infty u(U-u)\,dy = \frac{\tau_0}{\rho}$$

$$\int_0^\infty u(U-u)\,dy = U^2\int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1-\frac{2y}{\delta}+\frac{y^2}{\delta^2}\right) dy$$

$$\left(\int_0^\infty = \int_0^\delta + \int_\delta^\infty\right)$$

$$\frac{y}{\delta} = p$$

$$dy = dp\,\delta$$

$$= \delta U^2 \int_0^1 (2p-p^2)(1-2p+p^2)\,dp$$

$$= \delta U^2 \int_0^1 (2p - 4p^2 + 2p^3 - p^2 + 2p^3 - p^4)\,dp$$

$$= \delta U^2 \left(\frac{2p^2}{2} - \frac{4p^3}{3} + \frac{2p^4}{4} - \frac{p^3}{3} + \frac{2p^4}{4} - \frac{p^5}{5}\right)\Big|_0^1$$

$$= \delta U^2 \left(1 - \frac{4}{3} + \frac{1}{2} - \frac{1}{3} + \frac{1}{2} - \frac{1}{5}\right)$$

$$= \delta U^2 \left(2 - \frac{1}{5} - \frac{5}{3}\right) = \delta U^2\left(\frac{2}{15}\right)$$

$$\text{L.H.S.} = \frac{d}{dx}\left(\delta(U^2)\times\frac{2}{15}\right) = \boxed{\frac{2U^2}{15}\frac{d\delta}{dx}}$$

$$\text{R.H.S.} = \frac{\tau_0}{\rho} = \nu\left(\frac{\partial u}{\partial y}\right)\Big|_{y=0}$$

$$= \nu U\left(\frac{2}{\delta} - \frac{2yU}{\delta^2}\right)\Big|_{y=0}$$

$$= \frac{2U\nu}{\delta}$$

now L.H.S. = R.H.S.

$$\frac{2U^2}{15}\frac{\partial\delta}{dx} = \frac{2U\nu}{\delta}$$

$$\int \delta\, d\delta = \int 15\left(\frac{\nu}{U}\right) dx$$

$$\frac{\delta^2}{2} = 15\frac{\nu}{U}x$$

$$\delta = \sqrt{30}\sqrt{\frac{\nu x}{U}}$$

$$\boxed{\delta = 5.4772\sqrt{\frac{\nu x}{U}}} \Rightarrow \boxed{\frac{\delta}{x} = \frac{5.4772}{\sqrt{Re}}}$$

Blasius $\Rightarrow$ $\delta = 5\sqrt{\frac{\nu x}{U}}$ cubic $\Rightarrow$ $\delta = 4.64\sqrt{\frac{\nu x}{U}}$

So δ for quadratic assumption is the highest among Blasius and cubic assumption.

momentum thickness (θ)

$$U^2\theta = \int_0^{\delta} u(U-u)\,dy$$

$$U^2\theta = \frac{2U^2}{15}\delta \quad ; \quad \theta = \frac{2\delta}{15} = \frac{2}{15}\sqrt{30}\sqrt{\frac{\nu x}{U}}$$

$$\frac{\theta}{x} = \frac{2\sqrt{30}}{15}\frac{1}{\sqrt{Re}} = \boxed{\frac{0.7303}{\sqrt{Re}}}$$

$$\Rightarrow \frac{\theta}{x} = \frac{0.664}{\sqrt{Re}} \quad ; \quad \text{cubic} \Rightarrow \frac{\theta}{x} = \frac{0.646}{\sqrt{Re}}$$

as expected from $\frac{\delta}{x}$; $\frac{\theta}{x}$ for quadratic is more than both Blasius and cubic

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