Coupled Problems: Homework

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1 Transmision conditions

1.1

The deflection of an Euler-Bernouilli beam is governed by the differential equation

$$\boldsymbol{EI\frac{d^4v}{dx^4} = f}$$

where EI is a mechanical property of the beam section and the beam material and f is the distributed load. Assuming for example that the beam is clamped at $x = 0$ and $x = L$, the principle of virtual work states that the solution $v(x)$ satisfies

$$\boldsymbol{EI\int_0^L \frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2} = \int_0^L \delta v\ f}$$

for all δv such that $\delta v(0) = \delta v(L)$, $\frac{d\delta v}{dx}(0) = \frac{d\delta v}{dx}(L) = 0$.

a)

Postulate the space of functions where both v and δv must belong.

From the PVW, it is evident that :

$$v \in \left\{ v(x) : \Omega \to \mathbb{R} \mid v(0) = v(1) = 0, \partial_x v(0) = \partial_x v(1) = 0, \int \left(\frac{d^2u}{dx}\right)^2 < \infty \right\}$$

$$\delta v \in \left\{ \delta v(x) : \Omega \to \mathbb{R} \mid \delta v(0) = \delta v(1) = 0 \int \left(\frac{d^2\delta u}{dx}\right)^2 < \infty \right\}$$

b)

If $[0, L] = [0, P] \cup (P, L]$, obtain transmision conditions at P implied by regularity requirements.

In order to achieve regularity, there cannot be no jumps for v and $\frac{dv}{dx}$ across interface P:

$$[\![v]\!]_P = 0 \to v_1(P) = v_2(P)$$
$$[\![\partial_x v]\!]_P = 0 \to \partial_x v_1(P) = \partial_x v_2(P)$$

c)

Obtain the transmission conditions at P that follow by imposing in the PVW that the integral is additive.

The integral in the PVW is additive and can be expressed as:

$$EI\int_0^P \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} + EI\int_P^L \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} = \int_0^P \delta v\ f + \int_P^L \delta v\ f \tag{1}$$

If we obtain the variational form at subdomain 1:

$$EI\int_0^P \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} - \left[EI\frac{d\delta v}{dx}\frac{d^2 v}{dx^2}\right]_P + \left[EI\delta v\frac{d^3 v}{dx^3}\right]_P = \int_0^P \delta v\ f$$

and subdomain 2:

$$EI\int_P^L \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} - \left[EI\frac{d\delta v}{dx}\frac{d^2 v}{dx^2}\right]_P + \left[EI\delta v\frac{d^3 v}{dx^3}\right]_P = \int_P^L \delta v\ f$$

Adding the two integrals and comparing with equation 1, we have:

$$\left[EI\frac{d\delta v_1}{dx}\frac{d^2 v_1}{dx^2}\right]_P - \left[EI\frac{d\delta v_2}{dx}\frac{d^2 v_2}{dx^2}\right]_P = 0$$

$$\left[EI\delta v\frac{d^3 v_1}{dx^3}\right]_P - \left[EI\delta v\frac{d^3 v_2}{dx^3}\right]_P = 0$$

1.2

The Maxwell problem consists in finding a vector field $\boldsymbol{u} : \boldsymbol{\Omega} \rightarrow \boldsymbol{\mathcal{R}}^3$ such that

$$\boldsymbol{\nu\nabla\times\nabla\times u = f \quad in\Omega}$$

$$\boldsymbol{\nabla\cdot u = 0 \quad in\Omega}$$

$$\boldsymbol{n\times u = 0 \quad on\partial\Omega}$$

where $\boldsymbol{\nu > 0}$, f is a divergence free force field and n the unit external normal. Equation $\boldsymbol{\nabla\cdot u = 0}$ is in fact redundant.

a)

Write a variational statement of the problem. Postulate the space of functions where u must blong. Justify the answer.

The variational statement of the problem is obtained premultiplying by a test function $\boldsymbol{w}$ and integrating:

$$-\int_\Omega \nu(\nabla x\boldsymbol{w}\cdot(\nabla\times(\nabla x\boldsymbol{u})) = \int_\Omega \boldsymbol{w\cdot f}$$

Operating:

$$-\nu\int_\Omega(\nabla\times\boldsymbol{w})\ :\ (\nabla\times\boldsymbol{u}) = \int_\Omega \boldsymbol{w\cdot f} - \nu\int_{\partial\Omega^N}(\boldsymbol{w}\cdot(n\times(\nabla\times\boldsymbol{u})$$

Since $\boldsymbol{n}\times\boldsymbol{u} = \boldsymbol{0}$, the Neumann term vanishes:

$$-\nu\int_\Omega(\nabla\times\boldsymbol{w})\cdot(\nabla\times\boldsymbol{u}) = \int_\Omega \boldsymbol{w\cdot f}$$

Thus, the space in which $\boldsymbol{u}$ must belong is:

$$\boldsymbol{u}\in\left\{\boldsymbol{u}:\Omega\rightarrow\mathbb{R}^3\cdot\boldsymbol{n\times u = 0}on\partial\Omega, \int(\boldsymbol{\nabla\times u})^2<\infty\right\}$$

b)

If Γ is a surface that intersects Ω, obtain the transmission conditions across this surface implied by regularity requirements.

From regularity requirements, the following must hold across Γ:

$$[\![\boldsymbol{n} \times \boldsymbol{u}]\!]_\Gamma = 0 \rightarrow v_1(P) = v_2(P)$$

c)

Obtain the transmission conditions across Γ that follow by imposing in the variational form of the problem that the integral is additive.

The additivity of the integral provides:

$$-\nu \int_{\Omega_1} (\nabla \times \boldsymbol{w}) \cdot (\nabla \times \boldsymbol{u}) - \nu \int_{\Omega_2} (\nabla \times \boldsymbol{w}) \cdot (\nabla \times \boldsymbol{u}) = \int_{\Omega_1} \boldsymbol{w} \cdot \boldsymbol{f} + \int_{\Omega_2} \boldsymbol{w} \cdot \boldsymbol{f}$$

However, if we analize both subdomains separately:

$$-\nu \int_{\Omega_1} (\nabla\times\boldsymbol{w})\cdot(\nabla\times\boldsymbol{u}) - \nu \int_{\Omega_2} (\nabla\times\boldsymbol{w})\cdot(\nabla\times\boldsymbol{u}) = \int_{\Omega_1} \boldsymbol{w} \cdot \boldsymbol{f} + \int_{\Omega_2} \boldsymbol{w} \cdot \boldsymbol{f} - \nu \int_{\Gamma_1} (\boldsymbol{w}\cdot(n\times(\nabla\times\boldsymbol{u}) - \nu \int_{\Gamma_2} (\boldsymbol{w}\cdot(n$$

Thus:

$$\nu \int_\Gamma (\boldsymbol{w} \cdot (n \times (\nabla \times \boldsymbol{u}_1) + n \times (\nabla \times \boldsymbol{u}_2)) = 0$$

Thus:

$$n \times (\nabla \times \boldsymbol{u}_1) + n \times (\nabla \times \boldsymbol{u}_2) = 0$$

1.3

The Navier equations for an elastic material can be written in three different ways:

$$\boldsymbol{-2\mu\nabla \cdot (\varepsilon(u)) - \lambda\nabla(\nabla \cdot u) = \rho b}$$
$$\boldsymbol{-\mu\triangle u - (\lambda + \mu)\nabla(\nabla \cdot u) = \rho b}$$
$$\boldsymbol{\mu\nabla \times (\nabla \times u) - (\lambda + 2\mu)\nabla(\nabla \cdot u) = \rho b}$$

Let us assume that $\boldsymbol{u} = \boldsymbol{0}$ on $\boldsymbol{\partial\Omega}$

a)

Write down the variational form of the previous equations in the appropiate functional spaces.

Integrating a premultiplying by a test function $\boldsymbol{w}$:

1.
$$-2\mu \int_\Omega \nabla \cdot (\varepsilon(\boldsymbol{u})) \cdot \boldsymbol{w} - \lambda \int_\Omega \nabla(\nabla \cdot \boldsymbol{u}) = \int_\Omega \rho \boldsymbol{b}$$

2.
$$-\mu \int_\Omega \triangle \boldsymbol{u} - \int_\Omega (\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) = \int_\Omega \rho \boldsymbol{b}$$

3.

$$\mu \int_{\Omega} \nabla \times (\nabla \times \boldsymbol{u}) - \int_{\Omega} (\lambda + 2\mu) \nabla (\nabla \cdot \boldsymbol{u}) = \int_{\Omega} \rho \boldsymbol{b}$$

Operating:

1.

$$2\mu \int_{\Omega} \nabla \boldsymbol{w} : (\nabla \varepsilon(\boldsymbol{u})) + \lambda \int_{\Omega} \nabla(\boldsymbol{w}) : (\nabla \boldsymbol{u}) = \int_{\Omega} \rho \boldsymbol{b} + 2\mu \int_{\partial\Omega^N} \varepsilon \boldsymbol{n} \cdot \boldsymbol{w} + \lambda \int_{\partial\Omega^N} \nabla \boldsymbol{u} \boldsymbol{n} \cdot \boldsymbol{w}$$

2.

$$(\lambda + 2\mu) \int_{\Omega} \nabla(\boldsymbol{w}) : (\nabla \boldsymbol{u}) = \int_{\Omega} \rho \boldsymbol{b} + (\lambda + 2\mu) \int_{\partial\Omega^N} \nabla \boldsymbol{u} \boldsymbol{n} \cdot \boldsymbol{w}$$

3.

$$-\mu \int_{\Omega} (\nabla \times \boldsymbol{w}) \cdot (\nabla \times \boldsymbol{u}) + (\lambda + 2\mu) \int_{\Omega} \nabla(\boldsymbol{w}) : (\nabla \boldsymbol{u}) =$$
$$= \int_{\Omega} \rho \boldsymbol{b} - \mu \int_{\partial\Omega^N} n \times (\nabla \times \boldsymbol{u}) \cdot \boldsymbol{w} + (\lambda + 2\mu) \int_{\partial\Omega^N} \nabla \boldsymbol{u} \boldsymbol{n} \cdot \boldsymbol{w}$$

Since we have Dirichlet boundary conditions at the whole boundary, we can get rid of the Neumann terms:

1.

$$2\mu \int_{\Omega} \nabla \boldsymbol{w} : (\nabla \varepsilon(\boldsymbol{u})) + \lambda \int_{\Omega} \nabla(\boldsymbol{w}) : (\nabla \boldsymbol{u}) = \int_{\Omega} \rho \boldsymbol{b}$$

2.

$$(\lambda + 2\mu) \int_{\Omega} \nabla(\boldsymbol{w}) : (\nabla \boldsymbol{u}) = \int_{\Omega} \rho \boldsymbol{b}$$

3.

$$-\mu \int_{\Omega} (\nabla \times \boldsymbol{w}) \cdot (\nabla \times \boldsymbol{u}) + (\lambda + 2\mu) \int_{\Omega} \nabla(\boldsymbol{w}) : (\nabla \boldsymbol{u}) = \int_{\Omega} \rho \boldsymbol{b}$$

b)

If Γ is a surface that intersects Ω, obtain the transmission conditions across Γ that follow by imposing in the variational form of the problem that the integral is additive.

The variational forms obtained are additivie. Thus:

1.

$$2\mu \int_{\Omega_1} \nabla \boldsymbol{w} : (\nabla \varepsilon(\boldsymbol{u})) + 2\mu \int_{\Omega_2} \nabla \boldsymbol{w} : (\nabla \varepsilon(\boldsymbol{u})) + \lambda \int_{\Omega_1} \nabla(\boldsymbol{w}) : (\nabla \boldsymbol{u}) +$$
$$+ \lambda \int_{\Omega_2} \nabla(\boldsymbol{w}) : (\nabla \boldsymbol{u}) = \int_{\Omega_1} \rho \boldsymbol{b} + \int_{\Omega_2} \rho \boldsymbol{b}$$

2.

$$(\lambda + 2\mu) \int_{\Omega_1} \nabla(\boldsymbol{w}) : (\nabla \boldsymbol{u}) + (\lambda + 2\mu) \int_{\Omega_2} \nabla(\boldsymbol{w}) : (\nabla \boldsymbol{u}) = \int_{\Omega_1} \rho \boldsymbol{b} + \int_{\Omega_2} \rho \boldsymbol{b}$$

3.

$$-\mu \int_{\Omega_1} (\nabla \times \boldsymbol{w}) \cdot (\nabla \times \boldsymbol{u}) - \mu \int_{\Omega_2} (\nabla \times \boldsymbol{w}) \cdot (\nabla \times \boldsymbol{u}) + (\lambda + 2\mu) \int_{\Omega_1} \nabla(\boldsymbol{w}) : (\nabla \boldsymbol{u}) +$$
$$+ (\lambda + 2\mu) \int_{\Omega_2} \nabla(\boldsymbol{w}) : (\nabla \boldsymbol{u}) = \int_{\Omega_1} \rho \boldsymbol{b} + \int_{\Omega_2} \rho \boldsymbol{b}$$

However, if we compare with the variational forms that would arise from obtaining the variational form at every subdomain, we get that the Neumann terms must be equal at Γ:

1.

$$2\mu\varepsilon(\boldsymbol{u_1})\boldsymbol{n} + \lambda \nabla \boldsymbol{u_1} \boldsymbol{n} = 2\mu\varepsilon(\boldsymbol{u_2})\boldsymbol{n} + \lambda \nabla \boldsymbol{u_2} \boldsymbol{n}$$

2.

$$\nabla \boldsymbol{u_1} \boldsymbol{n} = \nabla \boldsymbol{u_2} \boldsymbol{n}$$

3.

$$-\mu \boldsymbol{n} \times (\nabla \times \boldsymbol{u_1}) + (\lambda + 2\mu) \nabla \boldsymbol{u_1} \boldsymbol{n} = -\mu \boldsymbol{n} \times (\nabla \times \boldsymbol{u_2}) + (\lambda + 2\mu) \nabla \boldsymbol{u_2} \boldsymbol{n}$$

2 Domain decomposition methods

2.1

Consider Problem 1 of Section 1. Let $[0, L] = [0, L_1] \cup [L_2, L]$ with $L_2 < L_1$.

a)

Write down an iteration-by-subdomain based on a Schwarz additive domain decomposition methods.

Given u^0, $\partial_x u^0$, repeat for $k = 0, 1...$ until convergence:

1.

$$EI \frac{d^4 v}{dx^4} = f \quad in[0, L_1]$$
$$v_1^k = v_2^{k-1} \quad in[L_2, L_1]$$
$$\partial_x v_1^k = \partial_x v_2^{k-1} \quad in[L_2, L_1]$$
$$u_1^k = 0 \quad in x = 0$$

2.

$$EI \frac{d^4 v}{dx^4} = f \quad in[L_2, L]$$
$$v_2^k = v_1^{k-1} \quad in[L_2, L_1]$$
$$\partial_x v_2^k = \partial_x v_1^{k-1} \quad in[L_2, L_1]$$
$$u_2^k = 0 \quad in x = L$$

b)

Obtain the matrix version of the previous scheme once space has been discretized using finite elements.

The global problem can be written as:

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$$

Problems in $[0, L_1]$ and $[L_2, L]$ depend on matrices A_{11} and A_{22}. We define the restriction operator $\mathbb{R} : V_h \to V_h^i$, where V_h and V_h^i are the spaces of the global and local functions:

$$A_{11} = R_1 A R_1^T \quad A_{22} = R_1 2 A R_2^T \quad A_{1\Gamma} = R_1 A R_{\Gamma 1}^T \quad A_{2\Gamma} = R_2 A R_{\Gamma 2}^T$$

At every iteration we are solving two systems of equations in parallel:

$$A_{11} R_1 u_1^k = R_1 f_1 - A_{1\Gamma} R_{\Gamma 1} u_1^{k-1}$$

$$A_{22} R_2 u_2^k = R_2 f_2 - A_{2\Gamma} R_{\Gamma 2} u_2^{k-1}$$

2.1

Consider Problem 2 of Section 1. Let $\boldsymbol{\Gamma}$ be a surface that intersects $\boldsymbol{\Omega}$.

a)

Write down an iteration-by-subdomain based on the Dirichlet-Neumann coupling.

Given u^0 such that $\nabla \cdot u_0 = 0$, repeat for $k = 0, 1...$ until convergence:

1.
$$\nu \nabla \times \nabla \times \boldsymbol{u}_1^k = \boldsymbol{f} \quad in \quad \Omega$$
$$\boldsymbol{n} \times \boldsymbol{u}_1^k = 0 \quad on \quad \partial\Omega_1$$
$$\boldsymbol{n} \times \boldsymbol{u}_1^k = \boldsymbol{n} \times \boldsymbol{u}_2^{k-1} \quad on \quad \partial\Omega_1 \backslash \Gamma$$

2.
$$\nu \nabla \times \nabla \times \boldsymbol{u}_2^k = \boldsymbol{f} \quad in \quad \Omega$$
$$\boldsymbol{n} \times \boldsymbol{u}_2^k = 0 \quad on \quad \partial\Omega_2$$
$$\boldsymbol{n} \times \nabla \times \boldsymbol{u}_2^k = \boldsymbol{n} \times \nabla \times \boldsymbol{u}_1^k \quad on \quad \partial\Omega_1 \backslash \Gamma$$

b)

Obtain the matrix version of the previous scheme once space has been discretized using finite elements.

The global system of equations to be solved can be expressed in terms of inner nodes on Ω_1, Ω_2 and nodes in Γ:

$$\begin{bmatrix} A_{11} & 0 & A_{1\Gamma} \\ 0 & A_{22} & A_{2\Gamma} \\ A_{\Gamma 1} & A_{\Gamma 2} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_\Gamma \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_\Gamma \end{bmatrix}$$

Where U includes the unknowns of the problem $\boldsymbol{n} \times \boldsymbol{u}$ and $A_{\Gamma\Gamma} = A_{\Gamma\Gamma 1} + A_{\Gamma\Gamma 2}$. If we performe a Dirichlet correction in Ω_1, followed by a Neumann correction in Ω_2, at every iteration we must solve:

1. $$A_{11}U_1^k = F_1 - A_{1\Gamma}U_\Gamma^(k-1)$$

2. $$\begin{bmatrix} A_{22} & A_{2\Gamma} \\ A_{\Gamma 2} & A_{\Gamma\Gamma 2} \end{bmatrix} \begin{bmatrix} U_2^k \\ U_{\Gamma 2}^k \end{bmatrix} = \begin{bmatrix} F_2 \\ F_\Gamma - A_{\Gamma\Gamma 1}U_\Gamma^k - A_{\Gamma 1}U_1^k \end{bmatrix}$$

2.3

Consider the problem of finding $u : \Omega \to \mathbb{R}$ such that

$$\boldsymbol{-k\triangle u = f \quad in \quad \Omega}$$
$$\boldsymbol{u = 0 \quad on \quad \partial\Omega}$$

a)

Write down an iteration-by-subdomain based on the Dirichlet-Robin coupling.

Given u^0, repeat for $i = 0, 1...$ until convergence:

1. $$-k\triangle u_1^i = f_1 \quad in \quad \Omega_1$$
$$u_1^i = u_2^{i-1} \quad \partial\Omega\backslash\Gamma$$
$$u_1^i = 0 \quad on \quad \partial\Omega$$

2. $$-k\triangle u_2^i = f_2 \quad in \quad \Omega_2$$
$$\gamma u_2^i + k\frac{\partial u_2^i}{\partial n} = \gamma u_1^i + k\frac{\partial u_1^i}{\partial n} \quad on \quad \partial\Omega\backslash\Gamma$$
$$u_2^i = 0 \quad on \quad \partial\Omega$$

b)

Obtain the matrix version of the previous scheme once space has been discretized using finite elements.

Following the same approach as in section 2 b), we obtain that, at every iteration, we must solve the following systems of equations:

1. $$A_{11}U_1^i = F_1 - A_{1\Gamma}U_\Gamma^(i-1)$$

2. $$\begin{bmatrix} A_{22} & A_{2\Gamma} \\ (1+\gamma)A_{\Gamma 2} & A_{\Gamma\Gamma 2} \end{bmatrix} \begin{bmatrix} U_2^i \\ U_{\Gamma 2}^i \end{bmatrix} = \begin{bmatrix} F_2 \\ F_\Gamma - A_{\Gamma\Gamma 1}U_\Gamma^{i-1} - \gamma A_{\Gamma 1}U_1^i \end{bmatrix}$$

4 Monolithic and partitioned schemes in time

Consider the one-dimensional, transient, heat transfer equation:

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = f \quad in \quad [0,1]$$

$$u(x=0,t) = 0$$

$$u(x=1,t) = 0$$

$$u(x,t=0) = 0$$

4.1

Discretize it using the finite element method (linear elements, element size h) for the discretization in space, and a BDF1 scheme for the discretization in time. Write down the weak form of the problem and the resulting matrix form of the problem, including the corresponding boundary integrals if necessary. Consider $\boldsymbol{\kappa}$, f=1, $\boldsymbol{\delta t}$=1.

If we use the finite element method for spatial discretization:

$$(v, \partial_t u)_\Omega + \kappa\,(\partial_x v, \partial_x u)_\Omega = (v, f)_\Omega + \kappa\,\langle v, \partial_x u \rangle_{\Gamma^N}$$

Since no Neumann boundary conditions are imposed, we can remove Neumann's term. The algebraic version of the problem is:

$$M\frac{dU}{dt} + kU = F$$

If we use a BDF1 scheme for the discretization in time, we evaluate U at time step $n+1$ using:

$$\frac{dU}{dt} = \frac{U^{n+1} - U^n}{\Delta t}$$

Thus:

$$MU^{n+1} + \Delta t\, KU^{n+1} = \Delta t F + MU^n$$

If we use three linear elements with size h:

$$K = \begin{bmatrix} \frac{1}{h} & -\frac{1}{h} & 0 & 0 \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & 0 \\ 0 & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\ 0 & 0 & -\frac{1}{h} & \frac{1}{h} \end{bmatrix}$$

$$F = \begin{bmatrix} \frac{h}{2} \\ h \\ h \\ \frac{h}{2} \end{bmatrix}$$

4.2

Consider a domain decomposition approach for the previous problem. The left subdomain is composed of 2 elements (h=0.2), while the right subdomain is composed of 3 elements(h=0.2). Show that, if a monolithic approach is adopted, no boundary integrals are required at the interface. From now on, we denote the values at the nodes of the meshes as $u_0, u_1, u_2, u_3, u_4, u_5$. The interface is at u_2. If we adopt a monolithic approach, we are solving for the following problems: At left subdomain:

$$(v_1, \partial_t u_1)_\Omega + \kappa (\partial_x v_1, \partial_x u_1)_\Omega = (v_1, f_1)_\Omega + \kappa \langle v_1, \partial_x u_1 \rangle_\Gamma$$

At right subdomain:

$$(v_2, \partial_t u_2)_\Omega + \kappa (\partial_x v_2, \partial_x u_2)_\Omega = (v_2, f_2)_\Omega + \kappa \langle v_2, \partial_x u_2 \rangle_{\Gamma^N}$$

Since both grids match and we are using the same interpolation space for both v_1 and v_2, the system will become:

$$(v_2, \partial_t u_2)_\Omega + \kappa (\partial_x v_2, \partial_x u_2)_\Omega + (v_1, \partial_t u_1)_\Omega + \kappa (\partial_x v_1, \partial_x u_1)_\Omega = (v_2, f_2)_\Omega + (v_1, f_1)_\Omega$$

Thus, we do not have to integrate boundary terms.

4.3

Obtain the algebraic form of the Dirichlet-to-Neumann operator for the left subdomain, departing from given values of u_i^n at time step n, and an interface value u_2^{n+1}.

At the left subdomain we are solving a Dirichlet problem. We define $A = M + \Delta t K$, $b = F + MU^n$, $U_1 = [\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$ and $U_\Gamma = u_2$:

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_\Gamma^{n+1} \end{bmatrix} = \begin{bmatrix} b_1 - A_{1\Gamma} U_\Gamma^{n-1} \\ b_\Gamma \end{bmatrix}$$

We can get U_1 from first equation:

$$U_1^{n+1} = A_{11}^{-1} b_1$$

Substituting in second equation:

$$A_{\Gamma\Gamma} = b_\Gamma - A_{\Gamma 1} A_{11}^{-1} b_1$$

Thus, $S = A_{\Gamma\Gamma} = \frac{5}{h} = 12.5$

4.4

Obtain the algebraic form of the Neumann-to-Dirichlet operator for the right subdomain, departing from given values of u_i^n at time step n, and an interface value for the fluxes $\kappa \partial_x u^{n+1}$ at coordinate node 2. At the right subdomain we are imposing the flux. Algebraically it can be expressed as:

$$\begin{bmatrix} A_{22} & A_{2\Gamma} \\ A_{\Gamma 2} & A_{\Gamma\Gamma 2} \end{bmatrix} \begin{bmatrix} U_2^{n+1} \\ U_\Gamma^{n+1} \end{bmatrix} = \begin{bmatrix} b_2 \\ b_\Gamma - A_{\Gamma\Gamma 1} U_\Gamma^{n+1} - \gamma A_{\Gamma 1} U_1^{n+1} \end{bmatrix}$$

From first equation we have:

$$U_2 = A_{22}^{-1} b_2 - A_{22}^{-1} A_{2\Gamma} U_\Gamma$$

Thus, the second equation will be:

$$(A_{\Gamma\Gamma} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}) U_\Gamma^{n+1} = b_\Gamma - A_{\Gamma\Gamma} u_\Gamma - A_{\Gamma 1} u_1^{n+1} - A_{\Gamma 2} A_{22} b_2$$

Thus:

$$S = A_{\Gamma\Gamma} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}$$

4.5

Write down an iterative algorithm for a staggered approach applying Dirichlet boundary conditions at the interface to the left subdomain and Neumann boundary conditions at the interface for the right subdomain. Given the values at previous time step u^n solve:

1.
$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_\Gamma^{n+1} \end{bmatrix} = \begin{bmatrix} b_1 - A_{1\Gamma} U_\Gamma^{n-1} \\ b_\Gamma \end{bmatrix}$$

2.
$$\begin{bmatrix} A_{22} & A_{2\Gamma} \\ A_{\Gamma 2} & A_{\Gamma\Gamma 2} \end{bmatrix} \begin{bmatrix} U_2^{n+1} \\ U_\Gamma^{n+1} \end{bmatrix} = \begin{bmatrix} b_2 \\ b_\Gamma - A_{\Gamma\Gamma 1} U_\Gamma^{n+1} - \gamma A_{\Gamma 1} U_1^{n+1} \end{bmatrix}$$

4.6

Do the same for an iteration-by-subdomain scheme: Given the values at previous time step u^n, iterate for i=1,2..., and solve:

1.
$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} U_1^{i+1^n+1} \\ U_\Gamma^{i+1^n+1} \end{bmatrix} = \begin{bmatrix} b_1^i - A_{1\Gamma} U_\Gamma^{i^n} \\ b_\Gamma^i \end{bmatrix}$$

2.
$$\begin{bmatrix} A_{22} & A_{2\Gamma} \\ A_{\Gamma 2} & A_{\Gamma\Gamma 2} \end{bmatrix} \begin{bmatrix} U_2^{n+1} \\ U_\Gamma^{n+1} \end{bmatrix} = \begin{bmatrix} b_2^i \\ b_\Gamma^i - A_{\Gamma\Gamma 1} U_\Gamma^{i+1^n+1} - \gamma A_{\Gamma 1} U_1^{i+1^n+1} \end{bmatrix}$$

6 Fractional step methods

Consider the fractional step approach for the incompressible Navier-Stokes equations (Yosida scheme):

$$M \frac{1}{\delta t} (\tilde{U}^{n+1} - U^n) + K \tilde{U}^{n+1} = f - G \tilde{P}^{n+1}$$

$$D M^{-1} G P^{n+1} = \frac{1}{\delta t} D \tilde{U}^{n+1} - D M^{-1} G \tilde{P}^{n+1}$$

$$M \frac{1}{\delta t} (U^{n+1} - \tilde{U}^n) + \alpha K (U^{n+1} - \tilde{U}^{n+1}) + G(P^{n+1} - \tilde{P}^{n+1} = 0$$

6.1

Which is the optimal parameter for the α parameter? The optimal parameter for α seems to be 1. This way, we recover the slightly compressible Navier-Stokes equations when adding first and third equations and we reduce the error obtained for a given coupling of spatial and temporal discretizations.

6.2

What is the source of error of the scheme? The sources of error for this scheme may be:

1. Incompressibility is not enforced, it is a slightly compressible problem. Thus, the solutions obtained may be compressible and produce error when compared with solutions for incompressible Navier-Stokes equations.

2. The spatial and temporal discretizations (values of δt and mesh size, and order of the interpolations used for p and u.

3. The treatment of the non-linear term K.