# COUPLED PROBLEMS
## TASK- 1
## TRANSMISSION CONDITIONS

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$\boxed{1}$ (a) $$EI \int_0^L \frac{d^2 \delta v}{dx^2} \frac{d^2 v}{dx^2} = \int_0^L \delta v \, f \qquad \Omega = [0, L]$$

$$\int_0^L \delta v \, f < \infty \rightarrow \begin{array}{l} \delta v \in L_2(\Omega) \\ v \in L_2(\Omega) \end{array}$$

$$\int_0^L \frac{d^2 v}{dx^2} \frac{d^2 \delta v}{dx^2} < \infty \Rightarrow \begin{array}{l} \frac{d^2 v}{dx^2} \in L_2(\Omega) \\ \frac{d^2}{dx^2}(\delta v) \in L_2(\Omega) \end{array}$$

We know that

$$\delta v(0) = 0 \qquad \left.\frac{d}{dx}\delta v\right|_0 = 0$$

$$\delta v(L) = 0 \qquad \left.\frac{d}{dx}\delta v\right|_L = 0$$

$$\int_0^L \left(\frac{d\delta v}{dx}\right)^2 = \left[\frac{d\delta v}{dx}\delta v\right]_0^L - \int_0^\infty \frac{d^2\delta v}{dx^2}\delta v$$

$$= \left[\delta v(L) \left.\frac{d}{dx}(\delta v)\right|_L - \delta v(0) \left.\frac{d}{dx}(\delta v)\right|_0\right] - \int_0^L \frac{d^2}{dx^2}(\delta v)\,\delta v$$

$$= 0 - \int_0^L \frac{d^2}{dx^2}(\delta v)\,\delta v$$

Since $\delta v \in L_2(\Omega)$ & $\frac{d^2}{dx^2}(\delta v) \in L_2(\Omega)$

$$\frac{d}{dx}(\delta v) \in L_2(\Omega)$$

$$\Rightarrow \quad \delta v \in H^2(\Omega)$$

Since the beam is clamped

$$v(0) = 0 \qquad \left.\frac{dv}{dx}\right|_{x=L} = 0$$
$$v(L) = 0 \qquad \left.\frac{dv}{dx}\right|_{x=0} = 0$$

$$\Rightarrow v \in H^2(\Omega)$$

$$H^2(\Omega) = \{ u : \Omega \to R \mid u \in L_2(\Omega),\ \nabla u \in [L_2(\Omega)]^d,\ \nabla \cdot \nabla u \in L_2(\Omega) \}$$

(b) Transmission conditions implied by regularity requirement can be deduced by looking at the weak form. We saw in class that if $u \in H^1(\Omega)$, then it cannot be discontinuous (1) across a surface in 3D
(2) across a line/curve in 2D
(3) across a point in 1D

because the gradient in the weak form will not be defined across the interface. the condition thus was the continuity of the solution, jumps across the interface is zero

$$[\![ u ]\!] = 0$$

In our case $v \in H^2(\Omega)$, by similar logic we can deduce that the jumps of the derivatives across the interface is zero and also the jumps in the solution across the interface is zero

$$[\![ \frac{du}{dx} ]\!] = 0$$

$$[\![ v ]\!] = 0$$

(c) Let us find the weak form of the whole domain

$$EI \int_0^L \frac{d^4 v}{dx^4} \delta v = RHS = \int_0^L f_1 \delta v$$

$$= \left\{ \left[ \delta v \frac{d^3 v}{dx^3} \right]_0^L - \int_0^L \frac{d(\delta v)}{dx} \frac{d^3 v}{dx^3} \right\} EI$$

$$= \left\{ \left[ \delta v \frac{d^3 v}{dx^3} \right]_0^L - \left[ \frac{d}{dx}(\delta v) \frac{d^2 v}{dx^2} \right]_0^L + \int_0^L \frac{d^2(\delta v)}{dx^2} \frac{d^2 v}{dx^2} \right\} EI$$

But $\delta v(0) = \delta v(L) = 0$

$$\therefore \int_0^L \frac{d^2}{dx^2}(\delta v)\, \frac{d^2 v}{dx^2} = \int_0^L f_1\, \delta v \quad \text{——} ①$$

Finding the weak form in subdomains

$$E_1 I_1 \int_0^P \frac{d^4 v}{dx^4}\, \delta v = \int_0^P f_1\, \delta v = RHS_1$$

$$= \left\{ \left[ \delta v \frac{d^3 v}{dx^3} \right]_0^P - \left[ \frac{d}{dx}(\delta v)\, \frac{d^2 v}{dx^2} \right]_0^P + \int_0^P \frac{d^2}{dx^2}(\delta v)\, \frac{d^2 v}{dx^2} \right\} E_1 I_1$$

$$= \left\{ \left[ \delta v(P^-) \frac{d^3 v}{dx^3}\bigg|_{P^-} - \frac{d}{dx}(\delta v)\bigg|_{P^-} \frac{d^2 v}{dx^2}\bigg|_{P^-} \right] + \int_0^P \frac{d^2(\delta v)}{dx^2}\, \frac{d^2 v}{dx^2} \right\} E_1 I_1$$

Similarly

$$RHS_2 = \left\{ \left[ -\delta v(P^+) \frac{d^3 v}{dx^3}\bigg|_{P+} + \frac{d}{dx}(\delta v)\bigg|_{P+} \frac{d^2 v}{dx^2}\bigg|_{P+} \right] + \int_P^L \frac{d^2(\delta v)}{dx^2}\, \frac{d^2 v}{dx^2} \right\} E_2 I_2$$

Summing up, and comparing with ①, we obtain

$$E_1 I_1\, \delta v(P^-) \frac{d^3 v}{dx^3}\bigg|_{P^-} - E_2 I_2\, \delta v(P^+) \frac{d^3 v}{dx^3}\bigg|_{P+} = 0$$

and

$$E_2 I_2 \frac{d(\delta v)}{dx}\bigg|_{P+} \frac{d^2 v}{dx^2}\bigg|_{P+} - E_1 I_1 \frac{d(\delta v)}{dx}\bigg|_{P^-} \frac{d^2 v}{dx^2}\bigg|_{P^-} = 0$$

$$M = EI \frac{d^2 v}{dx^2} = \text{bending moment}$$

$$S = EI \frac{d^3 v}{dx^3} = \text{shear force}$$

$$M_1\, \delta v(P^-) - M_2\, \delta v(P^+) = 0$$

$$S_1 \frac{d(\delta v)}{dx}\bigg|_{P^-} - S_2 \frac{d(\delta v)}{dx}\bigg|_{P+} = 0$$

At interface, shear force & bending moment must be the same in both subdomains

$$\Rightarrow \quad [\![ \delta v ]\!]_P = 0$$

$$[\![ \frac{d}{dx}(\delta v) ]\!]_P = 0$$

$\boxed{2}$ (a) Given

$$\Rightarrow \nabla \times \nabla \times \underline{u} = \underline{f} \quad \text{in } \Omega$$
$$\nabla \cdot \underline{u} = 0 \quad \text{in } \Omega$$
$$\underline{n} \times \underline{u} = 0 \quad \text{on } \partial\Omega$$

Multiplying with a vector test function and integrating

$$\int_\Omega (\Rightarrow \nabla \times \nabla \times \underline{u}) \cdot \underline{v} = \int_\Omega \underline{f} \cdot \underline{v}$$

Using the vector identity

$$\nabla \cdot (\underline{a} \times \underline{b}) = (\underline{\nabla} \times \underline{a}) \cdot \underline{b} - (\underline{\nabla} \times \underline{b}) \cdot \underline{a}$$

Put $\underline{a} = \underline{\nabla} \times \underline{u}$

$b = \underline{v}$

$$\Rightarrow \int_\Omega \nabla \cdot ((\nabla \times \underline{u}) \times \underline{v}) + (\underline{\nabla} \times \underline{v}) \cdot (\underline{\nabla} \times \underline{u}) = \int_\Omega \underline{f} \cdot \underline{v}$$

Using divergence theorem for the 1st term on the left

$$\Rightarrow \int_{\partial\Omega} \underline{n} \cdot [(\nabla \times \underline{u}) \times \underline{v}] + \Rightarrow \int_\Omega (\nabla \times \underline{v}) \cdot (\nabla \times \underline{u}) = \int_\Omega \underline{f} \cdot \underline{v}$$

Using the propeety of scalar triple product

$$\underline{a} \cdot [\underline{b} \times \underline{c}] = [\underline{a} \times \underline{b}] \cdot \underline{c}$$

$$\Rightarrow \int_{\partial\Omega} [n \times (\nabla \times \underline{u})] \cdot \underline{v} + \Rightarrow \int_\Omega (\nabla \times \underline{v}) \cdot (\nabla \times \underline{u}) = \int_\Omega \underline{f} \cdot v$$

Rearranging

$$\Rightarrow \int_\Omega (\nabla \times \underline{u}) \cdot (\nabla \times \underline{v}) = \int_\Omega \underline{f} \cdot \underline{v} - \int_{\partial\Omega} [n \times (\nabla \times \underline{u})] \cdot \underline{v}$$

For solution $u$ to exist

$$\underline{u} \in [L_2(\Omega)]^d$$
$$\nabla \times \underline{u} \in [L_2(\Omega)]^d$$

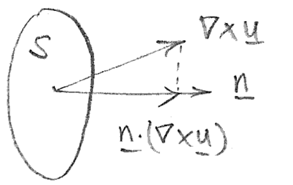
Defining $H(\mathrm{curl}) = \{ \underline{u} : \Omega \to \mathbb{R}^3 \mid \underline{u} \in [L_2(\Omega)]^d$
$\nabla \times \underline{u} \in [L_2(\Omega)]^d \}$

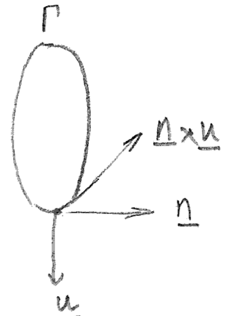
the space

$Y(0) = \{ \underline{u} \in H(\mathrm{curl}) : \underline{n} \times \underline{u} = 0 \text{ on } \partial\Omega \}$
(B.C)

Hence $\underline{u} \in Y(0)$
$\underline{v} \in Y(0)$

(b) Transmission condition by regularity requirements




Stokes theorem says

$$\int_S \underline{n} \cdot (\nabla \times \underline{u}) = \int_\Gamma \underline{n} \times \underline{u}$$

$\Rightarrow$ jump in the tangential component across the interface $= 0$

$[\![ \underline{n} \times \underline{u} ]\!] = 0$

(c) Transmission condition using the fact that integrals are additive. For each subdomain, the equations in weak form are.

$$\int_{\Omega_1} \nu_1 (\nabla \times \underline{u}_1) \cdot (\nabla \times \underline{v}) = \int_{\Omega_1} \underline{f}\, \underline{v} - \int_{\partial\Omega_1} [\underline{n}_1 \times (\nabla \times \underline{u}_1)] \cdot \underline{v}$$

$$\int_{\Omega_2} \nu_2 (\nabla \times \underline{u}_2) \cdot (\nabla \times \underline{v}) = \int_{\Omega_2} \underline{f}\, \underline{v} - \int_{\partial\Omega_2} [\underline{n}_2 \times (\nabla \times \underline{u}_2)] \cdot \underline{v}$$

Adding & using the equation for the whole domain

$$\int_\Omega \nu (\nabla \times \underline{u}) (\nabla \times \underline{v}) = \int_\Omega \underline{f}\, \underline{v} - \int_{\partial\Omega} [\underline{n} \times (\nabla \times \underline{u})] \cdot \underline{v}$$

only the boundary terms remain

$$\int_{\partial\Omega} [\underline{n} \times (\nabla \times \underline{u})] \cdot \underline{v} = \int_{\partial\Omega_1} [\underline{n}_1 \times (\nabla \times \underline{u}_1)] \cdot \underline{v} + \int_{\partial\Omega_2} [\underline{n}_2 \times (\nabla \times \underline{u}_2)] \cdot \underline{v}$$

$$\Rightarrow \int_{\Gamma} [\underline{n}_1 \times (\nabla \times \underline{u}_1)] \underline{v} + \int_{\Gamma} [\underline{n}_2 \times (\nabla \times \underline{u}_2)] = 0$$

$\Rightarrow$ jump

$[\![ \underline{n} \times (\nabla \times \underline{u}) ]\!] = 0$ is the necessary transmission condition.

$\boxed{3}$ The N-S equations given are equivalent and we prove this first

Firstly we show ① $\to$ ②

$$-2\mu \, \nabla \cdot [\underline{\underline{\epsilon}}(\underline{u})] - \lambda \nabla (\nabla \cdot \underline{u}) = \rho \underline{b}$$

$$\underline{\underline{\epsilon}}(u) = \frac{\nabla u + \nabla u^T}{2}$$

$$\therefore -\frac{2\mu}{2} [\nabla \cdot \nabla u + \nabla \cdot \nabla u^T] - \lambda \nabla (\nabla \cdot \underline{u}) = \rho \underline{b}$$

$$\Rightarrow -\mu [\partial_j (\partial_j u_i) + \partial_j (\partial_i u_j)] - \lambda \partial_i (\partial_i u_i) = \rho \underline{b}$$

$$\Rightarrow -\mu [\nabla \cdot \nabla \underline{u} + \nabla (\nabla \cdot \underline{u})] - \lambda \nabla (\nabla \cdot \underline{u}) = \rho \underline{b}$$

$$-\mu [\nabla \cdot \nabla \underline{u}] - (\mu + \lambda) \nabla (\nabla \cdot \underline{u}) = \rho \underline{b}$$

$$-\mu \Delta \underline{u} - (\mu + \lambda) \nabla (\nabla \cdot \underline{u}) = \rho \underline{b}$$

Now we show ③ $\to$ ②

$$\mu \nabla \times (\nabla \times \underline{u}) - (\lambda + 2\mu) \nabla (\nabla \cdot \underline{u}) = \rho \underline{b}$$

Using the identity

$$\nabla \times \nabla \times \underline{u} = \nabla (\nabla \cdot \underline{u}) - \Delta \underline{u}$$

$$-\mu \, \Delta \underline{u} - (\lambda + \mu) \, \nabla(\nabla \cdot \underline{u}) = \rho \, \underline{b}$$

Now that we have shown the equivalence, we develop the variational form.

$$\int_{\Omega} (-\mu \, \Delta \underline{u}) \, \underline{v} - (\lambda + \mu) \int_{\Omega} \nabla(\nabla \cdot \underline{u}) \cdot \underline{v} = \int_{\Omega} \rho \, \underline{b} \cdot \underline{v}$$

$$-\mu \left[ -\int_{\Omega} \nabla \underline{u} : \nabla \underline{v} + \underbrace{\int_{\partial\Omega} \underline{n} \cdot (v \, \nabla \underline{u})}_{①} \right]$$

$$-(\lambda + \mu) \left[ -\int_{\Omega} (\nabla \cdot \underline{u})(\nabla \cdot \underline{v}) + \underbrace{\int_{\partial\Omega} v \cdot (\nabla \cdot \underline{u})}_{②} \right] = \int_{\Omega} \rho \, \underline{b} \cdot \underline{v}$$

Since $\underline{v} = \underline{0}$ on $\partial\Omega$, we obtain

$$+\mu \int_{\Omega} \nabla \underline{u} : \nabla \underline{v} + (\lambda + \mu) \int_{\Omega} (\nabla \cdot \underline{u})(\nabla \cdot \underline{v}) = \int_{\Omega} \rho \, \underline{b} \cdot \underline{v}$$

Hence for solution to exist

$$\int_{\Omega} \nabla \underline{u} : \nabla \underline{v} < \infty \quad \Rightarrow \quad \nabla \underline{u} \in L_2(\Omega)$$

$$\int_{\Omega} (\nabla \cdot \underline{u})(\nabla \cdot v) < \infty \quad \Rightarrow \quad \nabla \cdot \underline{u} \in L_2(\Omega)$$

$$\int_{\Omega} \rho \, (\underline{b} \cdot \underline{v}) < \infty \quad \Rightarrow \quad u \in L_2(\Omega)$$

Since $H(\mathrm{div})\Omega \subset H^1(\Omega)$

$u \in H^1(\Omega)$

(b) To obtain transmission conditions using the additive property of integrals, we do the same procedure as explained in the previous problem

The boundary terms and the jumps in their quantities become significant

Term ① in the addition of subdomain eqn leads to

$$\int_{\Gamma} \underline{n}_1 \cdot (\mu_1 \nabla u_1) + \int_{\Gamma} \underline{n}_2 \cdot (\mu_2 \nabla u_2) = 0$$

$$[\![ \underline{n} \cdot (\mu \nabla \underline{u}) ]\!] = 0$$

term ② in the addition of subdomain eqns leads to

$$\int_{\Gamma} [(\mu_1 + \lambda_1) \nabla \cdot \underline{u}_1] + \int_{\Gamma} [(\mu_2 + \lambda_2) \nabla \cdot \underline{u}_2] = 0$$

$$\Rightarrow \cancel{(\mu_1 + \lambda_1)} \quad [\![ (\mu + \lambda) \nabla \cdot \underline{u} ]\!] = 0$$

These are the two transmission conditions

# COUPLED PROBLEMS

## TASK - 2

## DOMAIN DECOMPOSITION METHODS

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**1** (a) Additive schwartz - Algebraic version

$$E_1 I_1 \frac{d^4 v_1^{(k)}}{dx^4} = f \quad \text{in } \Omega_1 = [0, L_1]$$

$$v_1^{(k)} = 0 \quad \text{on } x = 0$$

$$\frac{dv_1^{(k)}}{dx} = 0 \quad \text{on } x = 0$$

$$v_1^{(k)} = v_2^{(k-1)} \quad \text{on } x = L_1$$

$$\frac{dv_1^{(k)}}{dx} = \frac{dv_2^{(k-1)}}{dx} \quad \text{on } x = L_1$$

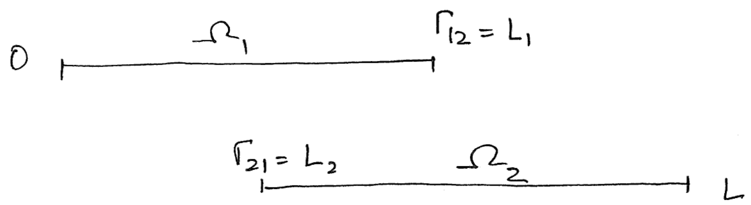
$$E_2 I_2 \frac{d^4 \cancel{x}_2^{(k)}}{dx^4} = f \quad \text{in } \Omega_2 = [L_2, L]$$

$$v_2^{(k)} = 0 \quad \text{on } L$$

$$\left.\frac{dv_2^{(k)}}{dx}\right|_{x=L} = 0 \quad \text{on } L$$

$$\left.\begin{aligned} v_2^{(k)} &= v_1^{(k-1)} \quad \text{on } x = L_2 \\ \frac{dv_2^{(k)}}{dx} &= \frac{dv_1^{(k-1)}}{dx} \quad \text{on } x = L_2 \end{aligned}\right\} \text{additive}$$

(b) Matrix version




For $\Omega_1$, $v$ & $\frac{dv}{dx}$ at $0$ & $\Gamma_{12}$ are known

$$\begin{bmatrix} A_{11} & A_{1L_1} \\ A_{L_1 1} & A_{L_1L_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_{L_1} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_{L_1} \end{bmatrix}$$

We find the value of interior $u_1$ because the value on the interface is known from $\Omega_2$, $u_{L_1}$ is known

$$\begin{bmatrix} A_{11} & A_{1L_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_{L_1} \end{bmatrix} = \begin{bmatrix} f_1 \\ \cancel{f_{L_1}} \end{bmatrix}$$

We assume that the b.c on the $x=0$ boundary is already imposed.

$$A_{11} u_1^{(k)} + A_{1L_1} u_{L_1}^{(k)} = f_1$$

$$u_1^{(k)} = A_{11}^{-1} \left( f_1 - A_{1L_1} u_{L_1}^{(k)} \right)$$

$\uparrow$

imposing b.c $u_{L_1}^{(k)} = u_{L_1}^{(k-1)}$ from $\Omega_2$

$$\therefore u_1^{(k)} = A_{11}^{-1} \left( f_1 - A_{1L_1} u_{L_1}^{(k-1)} \Big|_{\Omega_2} \right) \quad \text{—— (1)}$$

$$\begin{bmatrix} \breve{A}_{11} & \breve{A}_{1L_1} \\ \breve{A}_{L_1 1} & \breve{A}_{L_1L_1} \end{bmatrix} \begin{bmatrix} \frac{dv_1}{dx} \\ \frac{dv_{L_1}}{dx} \end{bmatrix} = \begin{bmatrix} \breve{f}_1 \\ \breve{f}_{L_1} \end{bmatrix}$$

from the previous procedure

$$\frac{dv_1}{dx}^{(k)} = \breve{A}_{11}^{-1} \left[ \breve{f}_1 - \breve{A}_{1L_1} \frac{dv_{L_1}}{dx}^{(k-1)} \Big|_{\Omega_2} \right] \quad \text{—— (2)}$$

(1) & (2) provide solution for the 4th order differential equation.

The same can be carried out in $\Omega_2$

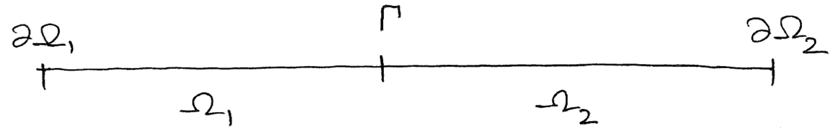
$$v_2^{(k)} = B_{11}^{-1} \left( g_1 - B_{1L_2} v_{L_2}^{(k-1)} \Big|_{\Omega_1} \right) \quad \text{—— (3)}$$

$$\frac{dv_2}{dx}^{(k)} = \breve{B}_{11}^{-1} \left( \breve{g}_1 - \breve{B}_{1L_2} \frac{dv_{L_2}}{dx}^{(k-1)} \Big|_{\Omega_1} \right) \quad \text{—— (4)}$$

(3) & (4) solve for solution in $\Omega_2$

$\boxed{2}$

$$\nu \, \nabla \times \nabla \times \underline{u} = \underline{f} \quad \text{in } \Omega$$
$$\nabla \cdot \underline{u} = 0 \quad \text{in } \Omega$$
$$\underline{n} \times \underline{u} = 0 \quad \text{on } \partial\Omega$$




(a) Dir-Neu coupling with iteration by subdomain

In $\Omega_1$

$$\nu_1 \nabla \times \nabla \times \underline{u}_1^{(k)} = f \quad \text{in } \Omega_1$$
$$\nabla \cdot \underline{u}_1^{(k)} = 0 \quad \text{in } \Omega_1$$
$$\underline{n}_1 \times \underline{u}_1^{(k)} = 0 \quad \text{on } \partial\Omega_1$$
$$n_1 \times (\nabla \times \underline{u}_1^{(k)}) = n_1 \times (\nabla \times \underline{u}_2^{(k-1)}) \quad \text{on } \Gamma \leftarrow \text{Neumann condition}$$

fixed n

In $\Omega_2$

$$\nu_2 \nabla \times \nabla \times \underline{u}_2^{(k)} = f \quad \text{in } \Omega_2$$
$$\nabla \cdot \underline{u}_2^{(k)} = 0 \quad \text{in } \Omega_2$$
$$n_2 \times \underline{u}_2^{(k)} = 0 \quad \text{on } \partial\Omega_2$$
$$\cancel{n_2 \times (\nabla \times \underline{u}_2) = n}$$
$$\underline{n}_2 \times \underline{u}_2^{(k)} = \underline{n}_2 \times \underline{u}_1^{(l)} \quad \text{on } \Gamma \leftarrow \text{dirichlet condition}$$

fixed n

If $l = (k-1)$ : Jacobi
$l = k$ : Gauss Siedel

(b) Here, the following relation holds for both subdomain

$$\nu \, \nabla \times \nabla \times u_i = f \quad \text{in } \Omega_i \quad , i = 1, 2$$
$$\underline{n}_i \times \underline{u}_i = 0 \quad \text{on } \partial\Omega_i \quad , i = 1, 2$$

$$\left.\begin{aligned} \underline{n} \times \underline{u}_1 &= \underline{n} \times \underline{u}_2 \\ \underline{n} \times (\nabla \times u_1) &= \underline{n} \times (\nabla \times u_2) \end{aligned}\right] \text{ on } \Gamma$$

for fixed 'n'

Direct method (Steklov - Poincaré operator)

$$u_i = \overset{\circ}{u}_i + \tilde{u}_i \quad , \quad i = 1, 2$$

$$\begin{aligned} \nu\, \nabla \times \nabla \times \underline{\overset{\circ}{u}}_i &= f && \text{in } \Omega_i \\ \underline{n} \times \underline{u}_i &= 0 && \text{on } \partial\Omega_i \\ \underline{n} \times \underline{\overset{\circ}{u}}_i &= 0 && \text{on } \Gamma \end{aligned}$$

and

$$\begin{aligned} \nu\, \nabla \times \nabla \times \tilde{u}_i &= 0 && \text{in } \Omega_i \\ \underline{n} \times \tilde{u}_i &= 0 && \text{on } \partial\Omega_i \\ \underline{n} \times \underline{\tilde{u}}_i &= \phi && \text{on } \Gamma \end{aligned}$$

We need to obtain $\phi$ such that $\underline{u}_i = \overset{\circ}{u}_i + \tilde{u}_i$ and the neumann condition holds

$$\underline{n} \times (\nabla \times u_1) = \underline{n} \times (\nabla \times u_2)$$

$$\underline{n} \times (\nabla \times (\overset{\circ}{u}_1 + \tilde{u}_1)) = \underline{n} \times (\nabla \times (\overset{\circ}{u}_2 + \tilde{u}_2))$$

$$\underline{n} \times (\nabla \times \underline{\overset{\circ}{u}}_1) - \underline{n} \times (\nabla \times \overset{\circ}{u}_2) = \underline{n} \times (\nabla \times \tilde{u}_2)) - \underline{n} \times (\nabla \times \tilde{u}_1)$$

$$\phi \rightarrow \underline{n} \times (\nabla \times \overset{\circ}{u}_1) - \underline{n} \times (\nabla \times \overset{\circ}{u}_2)$$

$\mathcal{S}$ is the steklov Poincaré's operator

$$G_\Gamma = \underline{n} \times (\nabla \times \tilde{u}_2) - n \times (\nabla \times \tilde{u}_1)$$

$$\mathcal{S}\phi = G_\Gamma$$

(c) Matrix version of the steklov - poincaré operator

$$\begin{bmatrix} A_{11} & A_{1\Gamma} & 0 \\ A_{\Gamma 1} & A_{\Gamma\Gamma} & A_{\Gamma 2} \\ 0 & A_{2\Gamma} & A_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_\Gamma \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_\Gamma \\ f_2 \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} u_1^{(k)} \\ u_\Gamma^{(k)} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_\Gamma - A_{\Gamma 2} u_2^{(k)} - A_{\Gamma\Gamma}^{(2)} u_\Gamma^{(k-1)} \end{bmatrix}$$

neumann condition.

we get $u_2^{(k)}$ from the eqn

$$A_{22} u_2^{(k)} = f_2 - A_{2\Gamma} u_\Gamma^{(k-1)}$$ ← dirichlet condition

$\boxed{3}$

$$-k \Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

(a) Iteration by subdomain with Dirichlet-Robin coupling

$$-k_1 \Delta u_1^{(k)} = f \quad \text{in } \Omega_1$$
$$u_1^{(k)} = 0 \quad \text{on } \partial\Omega_1$$
$$k_1 \frac{\partial u_1^{(k)}}{\partial n} + \tau_1 u_1^{(k)} = k_2 \frac{\partial u_2^{(k-1)}}{\partial n} + \tau_2 u_2^{(k-1)} \quad \text{on } \Gamma$$ ← robin condition

$$-k_2 \Delta u_2^{(k)} = f \quad \text{in } \Omega_2$$
$$u_2^{(k)} = 0 \quad \text{on } \partial\Omega_2$$
$$u_2^{(k)} = u_1^{(l)} \quad \text{on } \Gamma$$

$l = k-1$ : Jacobi method (additive)
$l = k$ : Gauss-Seidel (multiplicative)

(b) Matrix version

We start with the weak form in the subdomains to assemble the matrices.

In $\Omega_1$

$$\int_{\Omega_1} k_1 \nabla v \cdot \nabla u_1 - \int_{\partial\Omega_1 + \Gamma} k_1 v (\underline{n} \cdot \nabla u) = \int_{\Omega_1} f v$$

Since $u = 0$ on $\partial\Omega_1$, only boundary integral on $\Gamma$ remains and now we use the robin condition

$$k_1 (\underline{n} \cdot \nabla u) = k_1 \frac{\partial u_1}{\partial n} = k_2 \frac{\partial u_2}{\partial n} + \tau_2 u_2 - \tau_1 u_1$$

$$\int_{\Omega_1} k_1(\nabla v \cdot \nabla u_1) - \int_{\Gamma} v\left(k_2 \frac{\partial u_2}{\partial n} + T_2 u_2 - T_1 u_1\right) = \int_{\Omega} f v$$

$$k_1(\nabla u_1, \nabla v) - (k_2(\underline{n}\cdot\nabla u_2), v)_\Gamma - (T_2 u_2, v)_\Gamma - (T_1 u_1, v)_\Gamma = \langle f, v\rangle$$

Now we identify each of the integrals to the terms in the matrix

$$\begin{bmatrix} A_{11} & A_{1\Gamma} & 0 \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} + A_{\Gamma\Gamma}^{(2)} & A_{\Gamma 2} \\ 0 & A_{2\Gamma} & A_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_\Gamma \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_\Gamma \\ f_2 \end{bmatrix}$$

$k_1(\nabla u_1, v) \Rightarrow A_{11}$ internal contribution in $\Omega_1$

$-(T_1 u_1, v)_\Gamma \Rightarrow A_{\Gamma\Gamma}^{(1)}$ contribution of $u_1$ along the boundary $\Gamma$

$-(T_2 u_2, v)_\Gamma \Rightarrow A_{\Gamma 1}$ contribution of $u_2$ along the boundary $\Gamma$

$-(k_2 \underline{n}\cdot\nabla u_2, v)_\Gamma$

$$\therefore \begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} u_1^{(k)} \\ u_\Gamma^{(k)} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_\Gamma - A_{\Gamma\Gamma}^{(2)} u_\Gamma^{(k-1)} - A_{\Gamma 2} u_2^{(k-1)} \end{bmatrix}$$

In $\Omega_2$

$$\int_{\Omega_2} k_2(\nabla u_2 \cdot \nabla v) - \int_{\partial\Omega_2 + \Gamma} k_2 v\,(\underline{n}\cdot\nabla u_2) = \int_{\Omega_2} f v$$

- Since $u_2 = 0$ on $\partial\Omega_2$ and we impose the dirichlet boundary condition $u_2^{(k)} = u_1^{(k)}$ on $\Gamma$, we obtain the following matricial form

$$A_{22} u_2^{(k)} = f_2 - A_{2\Gamma} u_\Gamma^{(l)}$$

$l = k$ or
$l = k-1$

(c) Obtaining shur complement equation

We now use the matricial form previously developed and to obtain shur complement equation, the unknowns must be written interms of interface equations.

$\Omega_1$ :

$$A_{11} u_1^{(k)} + A_{1\Gamma} u_\Gamma^{(k)} = f_1$$

$$u_1^{(k)} = A_{11}^{-1}(f_1 - A_{1\Gamma} u_\Gamma^{(k)})$$

$$A_{\Gamma 1} u_1^{(k)} + A_{\Gamma\Gamma}^{(1)} u_\Gamma^{(k)} = f_\Gamma - A_{\Gamma 2} u_2^{(k-1)} - A_{\Gamma\Gamma}^{(2)} u_\Gamma^{(k-1)}$$

$$A_{\Gamma 1} A_{11}^{-1}(f_1 - A_{1\Gamma} u_\Gamma^{(k)}) + A_{\Gamma\Gamma}^{(1)} u_\Gamma^{(k)} = f_\Gamma - A_{\Gamma 2} u_2^{(k-1)} - A_{\Gamma\Gamma}^{(2)} u_\Gamma^{(k-1)}$$

$$[A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} + A_{\Gamma\Gamma}^{(1)}] u_\Gamma^{(k)} = (f_\Gamma - A_{\Gamma 2} u_2^{(k-1)} - A_{\Gamma\Gamma}^{(2)} u_\Gamma^{(k-1)} - A_{\Gamma 1} A_{11}^{-1} f_1) \quad \text{--- (1)}$$

$\Omega_2$ :

$$u_2^{(k)} = A_{22}^{-1}(f_2 - A_{2\Gamma} u_\Gamma^{(l)}) \quad \text{--- (2)}$$

Using (2) in (1)

$$[A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} + A_{\Gamma\Gamma}^{(1)}] u_\Gamma^{(k)} = [f_\Gamma - A_{\Gamma 2} A_{22}^{-1}(f_2 - A_{2\Gamma} u_\Gamma^{(l-1)}) - A_{\Gamma\Gamma}^{(2)} u_\Gamma^{(k-1)} - A_{\Gamma 1} A_{11}^{-1} f_1]$$

$$S_1 U_\Gamma^{(k)} = \tilde{G}$$

is the shur complement equation

(d) Consider Gauss siedel, $l = k$, the above equation becomes

$$[A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} + A_{\Gamma\Gamma}^{(1)}] u_\Gamma^{(k)} = f_\Gamma - A_{\Gamma 2} A_{22}^{-1} f_2 - A_{\Gamma 1} A_{11}^{-1} f_1 + A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma} u_\Gamma^{(k-1)} - A_{\Gamma\Gamma}^{(2)} u_\Gamma^{(k-1)}$$

The shur complement equation derived in class

$$S\,U^r = G$$

$$S = S_1 + S_2$$

$$= \left( A_{rr}^{(1)} - A_{r1} A_{11}^{-1} A_{1r} \right) + \left( A_{rr}^{(2)} - A_{r2} A_{22}^{-1} A_{2r} \right)$$

$$G = F_r - A_{r2} A_{22}^{-1} F_2 - A_{r1} A_{11}^{-1} F_1$$

Using this in the shur complement equation we derived

$$S_1 U_r^{(k)} = \tilde{G}$$

$$S_1 U_r^{(k)} = G - S_2 U_r^{(k-1)} \qquad S = S_1 + S_2$$

$$S_1 U_r^{(k)} = (S_1 - S) U_r^{(k-1)} + G$$

$$U_r^{(k)} = U_r^{(k-1)} + S_1^{-1} \left( G - S U_r^{(k-1)} \right)$$

this is Richardson with preconditioner $S_1^{-1}$

# COUPLED PROBLEMS
## TASK - 3
## COUPLING OF HETEROGENOUS PROBLEMS

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$\boxed{1}$ (a) In plane stress condition

$$\sigma_{13} = \sigma_{23} = \sigma_{33} = \sigma_{32} = \sigma_{31} = 0$$

the remaining stresses are given by

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix}$$

$$\epsilon_{ij} = \frac{1}{2}\left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

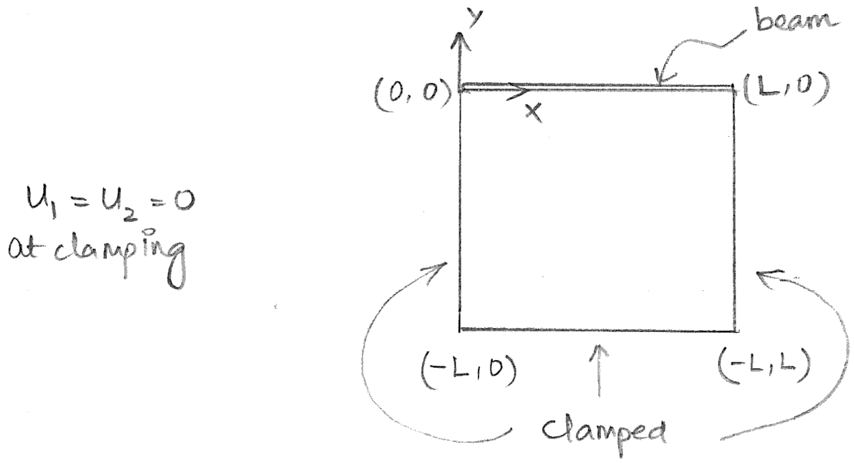
$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \epsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \epsilon_{12} = \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \frac{1}{2}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} \frac{\partial u_1}{\partial x_1} + \nu \frac{\partial u_2}{\partial x_2} \\ \nu \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \\ \frac{1-\nu}{2} \frac{\partial u_1}{\partial x_2} + \frac{1-\nu}{2} \frac{\partial u_2}{\partial x_1} \end{bmatrix}$$

Linear momentum equation states

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f} = 0$$

$$\frac{E}{1-\nu^2} \begin{bmatrix} \frac{\partial^2 u_1}{\partial x_1^2} + \nu \frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \left(\frac{1-\nu}{2}\right) \frac{\partial^2 u_1}{\partial x_2^2} + \left(\frac{1-\nu}{2}\right) \frac{\partial^2 u_2}{\partial x_1 \partial x_2} \\ \cancel{\left(\frac{1-\nu}{2}\right) \frac{\partial^2 u_1}{\partial x_1^2}} + \left(\frac{1-\nu}{2}\right) \frac{\partial^2 u_1}{\partial x_2 \partial x_1} + \left(\frac{1-\nu}{2}\right) \frac{\partial^2 u_2}{\partial x_1^2} + \nu \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_2^2} \\ 0 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ 0 \end{bmatrix} =$$




(b) Now taking into the consideration the effect of wall on the beam would be through the forcing term 'f' in the equation. This needs to be modified to include the effect of the wall

In the equation for the y component

$$EI \frac{d^4 v}{dx^4} = f_y$$

$$\& \; f_y = \underline{f}\big|_y = -(\nabla \cdot \underline{\underline{\sigma}})\big|_y$$

$$\therefore \; EI \frac{d^4 v}{dx^4} + (\nabla \cdot \underline{\underline{\sigma}})\big|_y = 0$$

(c) If instead of above, we use transmission conditions, the displacement (vertical) 'v' must be the same in the beam & the wall (regularity condition)

$$[\![ v ]\!]_{\Gamma} = 0$$

Now by the condition that the integrals must be additive, we obtain that the traction forces normal to the plane perpendicular to y-axis must be the same

$$[\![ \underline{t} ]\!]_{\Gamma} = [\![ \underline{n} \cdot (\nabla \cdot \underline{\underline{\sigma}}) ]\!]_{\Gamma} = 0$$

(d) The equation for the beam assumes zero horizontal displacement. Now we can talk in terms of two scenarios

(1) If we assume that the horizontal displacements are possible with the beam in perfect contact with the wall then $[\![ u ]\!]_\Gamma = 0$ and

$$[\![ \underline{t} \cdot (\nabla \cdot \underline{\underline{\sigma}}) ]\!] = 0$$

In this case Euler-Bernoulli beam theory is not valid anymore and components with $x$ deflection / displacement needs to be considered in the equation for the beam

(2) If we assume that horizontal displacements in the beam are impossible, then there must be a sliding contact between the beam and the wall. In this cace, jumps are possible

$[\![ u ]\!] \neq 0$ and

$$[\![ \underline{t} \cdot (\nabla \cdot \underline{\underline{\sigma}}) ]\!] \neq 0$$

Euler-Bernoulli model for the beam still prevails.

$\boxed{2}$ From the class notes we obtain the weak forms of stokes and darcy flows.

Stokes :

$$\int_{\Omega_S} \nu \nabla \delta u_s : \nabla u_s - \int_{\Omega_S} p_s \nabla \cdot \delta u_s - \int_{\partial\Omega_S} \delta u_s \cdot [n \cdot (-p_s I + \nu \nabla^s u_s)]$$

$$= \int_{\Omega} f \cdot \delta u_s$$

Darcy :

$$\int_{\Omega_D} \delta u_d \, k^{-1} u_D - \int_{\Omega_D} \phi \nabla \cdot \delta u_D + \int_{\partial\Omega_D} \phi \, n \cdot \delta u_D = 0$$

Equating that the normal component of velocities at the interface are equal

$$\Rightarrow \int_{\partial\Omega_S} \delta u_s \cdot [n \cdot (-p_s I + \nu \nabla^s u_s)] = \int_{\partial\Omega_D} \phi \, n \cdot \delta u_D$$

(a) Matricial version is as follows.

In $\Omega_S$

① $\int_{\Omega_S} \nu \nabla \delta u_S : \nabla u_S = A_S + A_S^{\Gamma}$ (velocity)

contributions in the internal + contributions on the interface

② $\int_{\Omega_S} p_S \nabla \cdot \delta u_S = B_S + B_S^{\Gamma}$ (pressure)

contributions in the internal + contributions on the interface

③ $\int_{\partial\Omega_S} \delta u_S (n \cdot - p_S \underline{\underline{I}}) = C_p^{\Gamma}$ (pressure)

contribution on the interface

④ $\int_{\partial\Omega_S} \delta u_S \cdot [n \cdot \nu \nabla^S u_S] = C_u^{\Gamma}$ (velocity)

contribution on the interface.

In $\Omega_D$

① $\int_{\Omega_D} \delta u_D \cdot K^{-1} u_D = A_D + A_D^{\Gamma}$ (velocity)

contribution in the internal + contribution on the interface

② $\int_{\Omega_D} \varphi \nabla \cdot \delta u_D = B_D + B_D^{\Gamma}$ (potential)

contribution in the internal + contribution on the interface

③ $\int_{\partial\Omega_D} \phi \; n \cdot \delta u_D = C_\phi^{\Gamma}$ (potential)

contribution on the interface.

The unknowns are $u_S$, $p$, $u_\Gamma = u_\lambda$, $u_D$ and $\phi$

Assembly looks like below ($C_u^\Gamma = C_\lambda^\Gamma$ $\because$ $u_\Gamma = u_\lambda$)

$$\begin{bmatrix} A_S & (B_S + C_p^\Gamma) & (C_\alpha^\Gamma + A_S^\Gamma) & 0 & 0 \\ B_S^T & 0 & (B_S^\Gamma)^T & 0 & 0 \\ 0 & 0 & A_D^\Gamma & A_D & (B_D + C_\varphi^\Gamma) \\ 0 & 0 & (B_D^\Gamma)^T & B_D^T & 0 \\ 0 & C_p^\Gamma & C_\lambda^\Gamma & 0 & -C_\varphi^\Gamma \end{bmatrix} \begin{bmatrix} u_S \\ p \\ u_\lambda \\ u_D \\ \varphi \end{bmatrix} = \begin{bmatrix} f_S \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The discrete form of equaling the interface values now is written as

$$C_p^\Gamma p + C_\lambda^\Gamma u_\lambda = C_\varphi^\Gamma \varphi$$

(b) Dir-Neu coupling iteration by subdomain

If we apply dir in $\Omega_S$

$$\begin{bmatrix} A_S & B_S \\ B_S^T & 0 \end{bmatrix} \begin{bmatrix} u_S^{(k)} \\ p^{(k)} \end{bmatrix} = \begin{bmatrix} f_S - C_p^\Gamma p^{(k-1)} \\ 0 - (B_S^\Gamma)^T u_\lambda^{(k-1)} \end{bmatrix}$$

We use the steklov-poincaré to transfer the values of $p$ & $u_\lambda$ using the eqn

$$C_p^\Gamma p^{(k)} + C_\lambda^\Gamma u_\lambda^{(k)} = C_\varphi^\Gamma \varphi^{(k)}$$

Now we use this as neumann condition in $\Omega_D$

$$\cancel{\begin{bmatrix} B_D^T & 0 \\ 0 & -C_\varphi^T \end{bmatrix} \begin{bmatrix} u_D \\ \varphi \end{bmatrix} = \begin{bmatrix} -(B_D^\Gamma)^T u_\lambda \end{bmatrix}}$$

$$\begin{bmatrix} A_D & B_D \\ B_D^T & 0 \end{bmatrix} \begin{bmatrix} u_D^{(k)} \\ \varphi^{(k)} \end{bmatrix} = \begin{bmatrix} -C_\varphi^\Gamma \varphi \\ 0 \end{bmatrix} = \begin{bmatrix} -[C_p^\Gamma p^{(\ell)} + C_\lambda^\Gamma u_\lambda^{(\ell)}] \\ 0 \end{bmatrix}$$

if $\ell = k-1$ : Jacobi (additive)
$\ell = k$ : Gauss Siedel (multiplicative)

# COUPLED PROBLEMS

## TASK - 4

## MONOLITHIC AND PARTITIONED SCHEMES in Time

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$\boxed{1}$ $\quad \frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = f$

$$u(0,t) = 0 \quad ; \quad u(1,t) = 0 \quad ; \quad u(x,0) = 0$$

To get the weak form, multiply with test function 'v'

$$\left(\frac{\partial u}{\partial t}, v\right) - k\left(\frac{\partial^2 u}{\partial x^2}, v\right) = (f, v)$$

$(\cdot,\cdot)$ is the L2 inner product

Using divergence theorem

$$(u_t, v) + k\left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right) - k \left\langle v, \underline{n} \cdot \frac{\partial u}{\partial x} \right\rangle = (f, v)$$

where $u, v \in V$, $V \in H_0^1(\Omega)$

Discretizing with element size $h$,

$$u \cong u_h = \sum_i u_i(t)\, N_i(x)$$

$$v :\simeq v_h = \sum_i v_i(t)\, N_i(x)$$

$$(u_{h,t}, v_h) + k(\nabla u_h, \nabla v_h) = (f, v_h) + k \langle v_h, \underline{n} \cdot \underline{\nabla} u_h \rangle$$

Applying BDF 1

$$\left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h\right) + k(\nabla u_h^{n+1}, \nabla v_h^{n+1}) = \langle f, v_h^{n+1} \rangle + k \langle v_h^{n+1}, \underline{n} \cdot \underline{\nabla} u_h^{n+1} \rangle$$

$$\left(\frac{u_h^{n+1}, v_h^{n+1}}{\Delta t}\right) + k(\nabla u_h^{n+1}, \nabla v_h^{n+1}) = (f, v_h^{n+1}) + k \langle v_h^{n+1}, \underline{n} \cdot \nabla u_h^{n+1} \rangle + \frac{(u_h^n, v_h^{n+1})}{\Delta t}$$

Applying the space discretization and defining the following matrices and substituting $f = k = 1$,

$$\Delta t = 1$$

$\underline{\underline{M}} := M_{ij} = \int_{\Omega} N_i N_j \, d\Omega$ mass matrix

$\underline{\underline{K}} := K_{ij} = \int_{\Omega} \underline{\nabla} N_j \cdot \nabla N_i \, d\Omega$ stiffness matrix

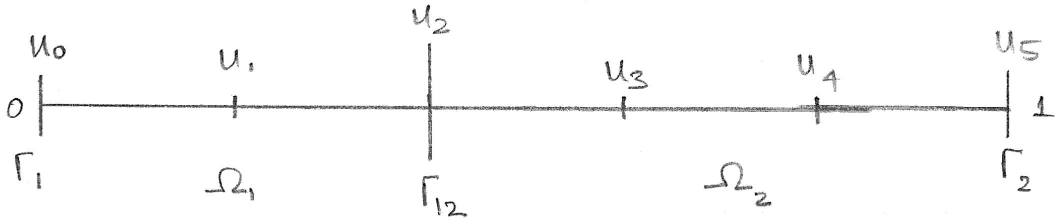
$\underline{F} := F_j = \int_{\Omega} f N_j \, d\Omega$ source vector

$\underline{\underline{q}} := q_{ij} = \int_{\Gamma} N_j (\underline{n} \cdot \underline{\nabla} N_i) \, d\Gamma$ boundary terms

$$M U^{n+1} + K U^{n+1} = F^{n+1} + M U^n + q$$

This is the discrete form of the equation.

$\boxed{2}$




$h = 0.2$

For domain $\Omega_1$, boundary $= \Gamma_1 + \Gamma_{12}$

$u(\Gamma_1) = 0$

Discrete equation is given as

$$\left(\frac{u_n^{n+1}}{\Delta t}, v_n\right) + k\left(\nabla u_n^{n+1}, \nabla v_n\right) - k \langle v_n, \underline{n}_1 \cdot \nabla u_n^{n+1} \rangle_{\Gamma_{12}} = (f^{n+1}, v_n) + \left(\frac{u_n^n}{\Delta t}, v_n\right) \quad \text{— (1)}$$

Similarly in $\Omega_2$

$$\left(\frac{u_n^{n+1}}{\Delta t}, v_n\right) + k\left(\nabla u_n^{n+1}, \nabla v_n\right) - k \langle v_n, \underline{n}_2 \cdot \nabla u_n^{n+1} \rangle_{\Gamma_{12}} = (f^{n+1}, v_n) + \left(\frac{u_n^n}{\Delta t}, v_n\right) \quad \text{— (2)}$$

Since the grids match and we use the same interpolation space $V_h$, we can now use the second transmission condition to our benefit.

$$\underline{n}_1 \cdot \underline{\nabla} u_{h1} + \underline{n}_2 \cdot \underline{\nabla} u_{h2} = 0$$

$$\therefore \langle v_n, n_1 \cdot \nabla u_n^{n+1} \rangle + \langle v_n, n_2 \cdot \nabla u_n^{n+1} \rangle = 0$$

Using this in the equation ① + ②

$$\left\{ \begin{array}{l} \left(\frac{u_n^{n+1}}{\Delta t}, v_n\right)_{\Omega_1} + k\left(\nabla u_n^{n+1}, \nabla v_n^{n+1}\right)_{\Omega_1} \\ + \left(\frac{u_n}{\Delta t}, v_n\right)_{\Omega_2} + k\left(\nabla u_n^{n+1}, \nabla v_n^{n+1}\right)_{\Omega_2} \end{array} \right\} = \left\{ \begin{array}{l} (f, v_n)_{\Omega_1} + \left(\frac{u_n^{n+1}}{\Delta t}, v_n\right)_{\Omega_1} \\ + (f, v_n)_{\Omega_2} + \left(\frac{u_n^{n+1}}{\Delta t}, v_n\right)_{\Omega_2} \end{array} \right\}$$

Hence interface terms vanish in a monolithic approach. Here $\Omega = \Omega_1 \oplus \Omega_2$

$\boxed{3}$ We use the following notation

$$U_1 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \quad U_\Gamma = [u_2], \quad U_2 = \begin{bmatrix} u_3 \\ u_4 \\ u_5 \end{bmatrix}$$

We need to solve for the left domain
Dir - Neu operator is to be used.

Values at time step 'n' are known $\Rightarrow$ $U_1^n, U_\Gamma^n, U_2^n$ are known

Interface value $u_2^{n+1}$ is known $\Rightarrow$ $U_\Gamma^{n+1}$ is known.

The question is to find $U_1^{n+1}$

The problem in geometric version is written as follows.
(Note: $u_1$ & $u_2$ here do not denote the nodal values but the solution in $\Omega_1$ & $\Omega_2$ respectively)

IC → initial condition

- In $\Omega_2$

$$\partial_t u_2 + \mathcal{L} u_2^{(n+1)} = f \quad \text{in } \Omega_2$$

$$u_2^{(n+1)} = \bar{u}|_{\Gamma_2} \quad \text{on } \Gamma_2 + (IC)$$

$$k_2 \frac{\partial u_2^{(n+1)}}{\partial n} = k_1 \frac{\partial u_1}{\partial n} \quad \text{on } \Gamma_{12}$$

Solve for $u_2^{(n+1)}$ — use in

In $\Omega_1$

$$\partial_t u_1 + \mathcal{L} u_1^{(n+1)} = f \quad \text{in } \Omega_1$$

$$u_1^{(n+1)} = \bar{u}|_{\Gamma_1} \quad \text{on } \Gamma_1 + (IC)$$

$$u_1^{(n+1)} = u_2^{(n+1)}$$

$$\Rightarrow u_1^{n+1} = u_{\Gamma_{12}} \quad \text{on } \Gamma_{12}$$

given value

In Algebraic form

$$\underbrace{\left(\frac{M}{\Delta t} + K\right)}_{A} U^{n+1} = \underbrace{f + \left(\frac{M}{\Delta t}\right) U^{n}}_{\breve{F} \,=\, \text{known}}$$

$$A U^{n+1} = \breve{F}$$

$$\begin{bmatrix} A_{11} & A_{1r} & 0 \\ A_{r1} & A_{rr} & A_{r2} \\ 0 & A_{2r} & A_{22} \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_r^{n+1} \\ U_2^{n+1} \end{bmatrix} = \begin{bmatrix} \breve{F}_1 \\ \breve{F}_r \\ \breve{F}_2 \end{bmatrix}$$

$\breve{F}$ is known, $U_r^{n+1}$ is known

$$\therefore \quad U_1^{n+1} = A_{11}^{-1} \left( \breve{F}_1 - A_{1r} U_r^{n+1} \right)$$ dirichlet problem.

$\boxed{4}$ In this case
We need to solve for the right domain
Neu - Dir operator is to be used

Values at time step 'n' are known $\Rightarrow$ $U_1^n, U_r^n, U_2^n$ are known

Value of flux $\phi^{n+1} = k \dfrac{\partial u^{n+1}}{\partial x}$ are known

The question is to find $U_2^{n+1}$

The problem in geometric form is written as follows.
( $u_1$ & $u_2$ denote solution in $\Omega_1$ & $\Omega_2$ )

In $\Omega_1$

$$\partial_t u_1 + \mathcal{L} u_1^{(n+1)} = f \quad \text{in } \Omega_1$$
$$u_1^{(n+1)} = u|_{\Gamma_1} \quad \text{on } \Gamma_1$$
$$u_1^{(n+1)} = u_2^{(n)} \quad \text{on } \Gamma_{12}$$

Solve for $u_1^{n+1}$ $\xrightarrow{\text{use it}}$

In $\Omega_2$

$$\partial_t u_2 + \mathcal{L} u_2^{(n+1)} = f \quad \text{in } \Omega_2$$
$$u_2^{(n+1)} = u|_{\Gamma_2} \quad \text{on } \Gamma_2$$
$$k_2 \frac{\partial u_2^{(n+1)}}{\partial n} = \phi \quad \text{on } \Gamma_{12}$$
$$= k_1 \frac{\partial u_1^{(n+1)}}{\partial n}$$

In algebraic terms

$$\begin{bmatrix} A_{11} & A_{1\Gamma} & 0 \\ A_{\Gamma 1} & A_{\Gamma\Gamma} & A_{\Gamma 2} \\ 0 & A_{2\Gamma} & A_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_\Gamma \\ U_2 \end{bmatrix}^{n+1} = \begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_\Gamma \\ \tilde{F}_2 \end{bmatrix}$$

$\tilde{F}$ is known

$$U_1^{n+1} = A_{11}^{-1} \left( \tilde{F}_1 - A_{1\Gamma} U_\Gamma^{n} \right)$$

$$\begin{bmatrix} A_{\Gamma\Gamma}^{(2)} & A_{\Gamma 2} \\ A_{2\Gamma} & A_{22} \end{bmatrix} \begin{bmatrix} U_\Gamma \\ U_2 \end{bmatrix}^{n+1} = \begin{bmatrix} \tilde{F}_\Gamma - A_{\Gamma 1} U_1^{n+1} - \overbrace{A_{\Gamma\Gamma}^{(1)} U_\Gamma^{n}}^{\phi} \\ F_2 \end{bmatrix} \leftarrow \text{neumann problem}$$

5 Staggered approch (subdomains can be solved in parallel)
we apply Dir B.C to left domain and Neu-B C to the right.

In $\Omega_1$

$$\partial_t u_1^{(n+1)} + \mathcal{L} u_1^{(n+1)} = f_1 \quad \text{in } \Omega_1$$

$$u_1^{(n+1)} = 0 \quad \text{on } \Gamma_1 \text{ (zero dirichet on the boundary } \Gamma_1)$$

$$u_1^{(n+1)} = u_2^{(n)} \quad \text{on } \Gamma_{12} \text{ (value on interface from the previous time step value of } u_2)$$

In $\Omega_1$

$$\partial_t u_2^{(n+1)} + \mathcal{L} u_2^{(n+1)} = f_2 \quad \text{in } \Omega_1$$

$$u_2^{(n+1)} = 0 \quad \text{on } \Gamma_2 \text{ (zero dirichlet on boundary } \Gamma_2)$$

$$k_2 \frac{\partial u_2^{(n+1)}}{\partial n} = k_1 \frac{\partial u_2^{(n)}}{\partial n} \quad \text{on } \Gamma_{12} \text{ (value at the interface from flux value of the previous time step)}$$

Solve above for $u_1^{(n+1)}$ & $u_2^{(n+1)}$ parallely.
Note that if n is the first time step, we apply initial condition in addition.

6 Substitution approach (subdomains are solved sequentially)

In $\Omega_1$

$$\partial_t u_1^{(n+1)} + \mathcal{L} u_1^{(n+1)} = f_1 \quad \text{in } \Omega_1$$

$$u_1^{(n+1)} = 0 \quad \text{on } \Gamma_1$$

$$u_1^{(n+1)} = u_2^{(n)} \quad \text{on } \Gamma_{12}$$

Solve for $u_1^{(n+1)}$ and use in the following

In $\Omega_2$

$$\partial_t u_2^{(n+1)} + \mathcal{L} u_2^{(n+1)} = f \quad \text{in } \Omega_2$$
$$u_2^{(n+1)} = 0 \quad \text{on } \Gamma_2$$
$$K_2 \frac{\partial u_2^{(n+1)}}{\partial n} = K_1 \frac{\partial u_1^{(n+1)}}{\partial n} \quad \text{on } \Gamma_{12}$$

Now we solve for $u_2^{(n+1)}$

In Iteration by sub-domain scheme; we apply an iteration counter to the substitution scheme and iterate until some convergence criteria is satisfied.

In $\Omega_1$

$$\partial_t u_1^{(n+1),i} + \mathcal{L} u_1^{(n+1),i} = f \quad \text{in } \Omega_1$$
$$u_1^{(n+1),i} = 0 \quad \text{on } \Gamma_1$$
$$u_1^{(n+1),i} = u_2^{(n),i} \quad \text{on } \Gamma_{12}$$

Solve for $u_1^{(n+1),i}$

In $\Omega_2$

$$\partial_t u_2^{(n+1),i} + \mathcal{L} u_2^{(n+1),i} = f \quad \text{in } \Omega_2$$
$$u_2^{(n+1),i} = 0 \quad \text{on } \Gamma_2$$
$$k_2 \frac{\partial u_2^{(n+1),i}}{\partial n} = k_1 \frac{\partial u_1^{(n+1),i}}{\partial n} \quad \text{on } \Gamma_{12}$$

Solve for $u_2^{(n+1),i}$

Update $u_2^{(n+1),i}$ & $u_1^{(n+1),i}$

Solve until
$$\frac{\| u_2^{(n+1),i+1} - u_2^{(n+1),i} \|_{L_2}}{\| u_2^{(n+1),i} \|_{L_2}} < \text{tolerance } (\epsilon_2)$$

and
$$\frac{\| u_1^{(n+1),i+1} - u_1^{(n+1),i} \|_{L_2}}{\| u_2^{(n+1),i} \|_{L_2}} < \text{tolerance } (\epsilon_1)$$

are satisfied.

CORRECTION.

$\boxed{3}$ In this I applied dirichlet in $\Omega_1$ & neumann in $\Omega_2$. We were asked to apply Dir-Neu which means it should be the other way around.

In $\Omega_1$

$$\partial_t u_1 + \mathcal{L} u_1^{(n+1)} = f \quad \text{in} \quad \Omega_1$$
$$u_1^{(n+1)} = \bar{u}\,|_{\Gamma_1} \quad \text{on} \quad \Gamma_1 + (IC)$$
$$k_1 \frac{\partial u_1^{(n+1)}}{\partial n} = k_2 \frac{\partial u_2^{(n)}}{\partial n} \quad \text{on} \quad \Gamma_{12}$$

In $\Omega_2$

$$\partial_t u_2 + \mathcal{L} u_2^{(n+1)} = f \quad \text{in} \quad \Omega_2$$
$$u_2^{(n+1)} = \bar{u}\,|_{\Gamma_2} \quad \text{on} \quad \Gamma_2 + (IC)$$
$$u_2^{(n+1)} = u_1^{(n)}| \quad \text{on} \quad \Gamma_{12}$$

In algebraic terms

$$\begin{bmatrix} A_{11} & A_{1r} & 0 \\ A_{r1} & A_{rr} & A_{r2} \\ 0 & A_{2r} & A_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_r \\ U_2 \end{bmatrix}^{n+1} = \begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_r \\ \tilde{F}_2 \end{bmatrix}$$

$\tilde{F}_r$ is known, $U_r^{n+1}$ is known

$$U_2^{n+1} = A_{22}^{-1} \left( \tilde{F}_2 - A_{2r} U_r^{n+1} \right)$$ ← dirichlet condition.

Using this in the othe eqn

$$\cancel{A_{rr}^{(1)}} \quad \begin{bmatrix} A_{11} & A_{1r} \\ A_{r1} & A_{rr}^{(1)} \end{bmatrix} \begin{bmatrix} U_1 \\ U_r \end{bmatrix}^{n+1} = \begin{bmatrix} \tilde{F}_1 - \cancel{A_{f2}} \\ \tilde{F}_r - A_{r2} U_2^{(n+1)} - A_{rr}^{(2)} U_r^{(n+1)} \end{bmatrix}$$

↑ neumann condition

$\boxed{4}$ We made the same mistake of application of transmission condition (reverse application) in this problem too. the corrected version reads as follows.

In $\Omega_1$

$$\partial_t u_1^{(n+1)} + \mathcal{L} u_1^{(n+1)} = f \quad \text{in } \Omega_1$$
$$u_1^{(n+1)} = \bar{u}_1 \quad \text{on } \Gamma_1 + IC$$
$$u_1^{(n+1)} = u_2^{(n)} \quad \text{on } \Gamma_{12}$$

In $\Omega_2$

$$\partial_t u_2^{(n+1)} + \mathcal{L} u_2^{(n+1)} = f \quad \text{in } \Omega_2$$
$$u_2^{(n+1)} = \bar{u}_2 \quad \text{on } \Gamma_2 + (IC)$$
$$k_2 \frac{\partial u_2^{(n+1)}}{\partial n} = k_1 \frac{\partial u_1^{(n)}}{\partial n}$$

In algebraic terms

$$\begin{bmatrix} A_{11} & A_{1\Gamma} & 0 \\ A_{\Gamma 1} & A_{\Gamma\Gamma} & A_{\Gamma 2} \\ 0 & A_{2\Gamma} & A_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_\Gamma \\ U_2 \end{bmatrix} = \begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_\Gamma \\ \tilde{F}_2 \end{bmatrix}$$

$\tilde{F}$ is known

$$U_1^{(n+1)} = A_{11}^{-1} \left( \tilde{F}_1 - A_{1\Gamma} U_\Gamma^{(n)} \right) \leftarrow \text{dirichlet condition}$$

$$\begin{bmatrix} A_{\Gamma\Gamma}^{(2)} & A_{\Gamma 2} \\ A_{2\Gamma} & A_{22} \end{bmatrix} \begin{bmatrix} U_\Gamma \\ U_2 \end{bmatrix}^{n+1} = \begin{bmatrix} \tilde{F}_\Gamma - A_{\Gamma 1} U_1^{(n+1)} - A_{\Gamma\Gamma}^{(1)} U_\Gamma^{n} \\ F_2 \end{bmatrix} \leftarrow \text{neumann condition}$$

## 7 Nitsche's method

The weak form of the problem in $\Omega_1$ is given as follows

$$(\partial_t u_1, v) + (\nabla u_1, \nabla v) - (v, \underline{n}\cdot\nabla u)_{\Gamma_D} = (f, v)$$

Nitsche's method is an extension of penalty method to make the system symmetric and therefore consistency is ensured. It is as shown below

$$(\partial_t u_1, v) + (\nabla u_1, \nabla v) - (v, \underline{n}\cdot\nabla u_1)_{\Gamma_D} + \alpha(v, u_1)_{\Gamma_D} - (u_1, \underline{n}\cdot\nabla v)_{\Gamma_D} = (f, v) + \alpha(v, u_2)_{\Gamma_D} - (u_2, \underline{n}\cdot\nabla v)_{\Gamma_D}$$

Solve for $u_1$, $u_2$ is known (Interface condition $u_1^{(n+1)} = u_2^{(n)}$)

For the original system, algebraic form was

$$U_1^{(n+1)} = A_{11}^{-1}\left(\tilde{F}_1 - A_{1\Gamma} U_\Gamma^{(n)}\right)$$

Now with the Nitsche's method

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & (A_{\Gamma\Gamma} + \alpha M + N) \end{bmatrix} \begin{bmatrix} U_1 \\ U_\Gamma \end{bmatrix}^{(n+1)} = \begin{bmatrix} F_1 \\ F_\Gamma + \alpha M U_2^{(n)} + N U_2^{(n)} \end{bmatrix}$$

$\alpha M$ term is from the penalty addition
$N$ term is from the symmetric term

The advantage of Nitsche's method is that '$\alpha$' value does not have to be very high. Condition number of the system increases with $\alpha$, but not as much as the penalty method. Smaller '$\alpha$' values are required for ensuring convergence and boundary condition implementation. Hence the conditioning of the matrix is comparitively better.

# COUPLED PROBLEMS
## TASK - 5

## OPERATOR SPLITTING TECHNIQUES

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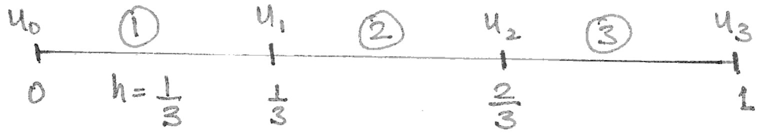
$\boxed{1}$ $$\frac{\partial u}{\partial t} + a_x \frac{\partial u}{\partial x} - k \frac{\partial^2 u}{\partial x^2} = f \quad \text{in } [0,1]$$

$$u(x=0,t) = 0$$
$$u(x=1,t) = 0$$
$$u(x,t=0) = 0$$

$$a_x = k = f = 1$$




$u(0,t)=0$ $\qquad$ $u(1,t)=0$

$u(x,0)=0$

Weak form

$$(u_t, v) + \left(\frac{du}{dx}, v\right) - \left(\frac{d^2u}{dx^2}, v\right) = \langle f, v \rangle$$

$$(u_t, v) + \left(\frac{du}{dx}, v\right) + \left(\frac{du}{dx}, \frac{dv}{dx}\right) - \left(v, n \cdot \frac{du}{dx}\right) = \langle 1, v \rangle \quad \text{—} (1)$$

$$u \simeq u_n = \sum_i u_i N_i(x) \qquad v \simeq v_n = \sum_j v_j N_j(x)$$

$$\frac{du_n}{dx} = \sum_i u_i \frac{dN_i(x)}{dx} \qquad V \subset H^1(\Omega)$$
$$V_0 \subset H_0^1(\Omega)$$

$$\frac{du_n}{dt} = \sum_i \frac{du_i}{dt} N_i(x) \qquad u \in V_0$$
$$v \in V_0$$

Substituting in ①

$$\underline{\underline{M}}\frac{d\underline{U}}{dt} + \underline{\underline{K}}\,\underline{U} + \underline{\underline{C}}\,\underline{U} = \underline{F}$$

$$\underline{\underline{M}} = \int_\Omega N_i N_j \, d\Omega \quad ; \quad \underline{\underline{K}} = \int_\Omega \frac{dN_i}{dx}\frac{dN_j}{dx}\, d\Omega$$

$$\underline{\underline{C}} = \int_\Omega N_j \frac{dN_i}{dx}\, d\Omega \quad ; \quad \underline{F} = \int_\Omega N_i(x)\, d\Omega$$

$\underline{U} \rightarrow$ vector of unknowns

Using BDF 1

$$\frac{\underline{\underline{M}}\,\underline{U}^{n+1}}{\Delta t} + \underline{\underline{K}}\,\underline{U}^{n+1} + \underline{\underline{C}}\,\underline{U}^{n+1} = \underline{F}^{n+1} + \frac{\underline{\underline{M}}\,\underline{U}^{n}}{\Delta t}$$

Now we need to find the quantities M, K, F, C in each element and assemble them into the global matrix. For demonstration, we carryout the procedure only for the first element

Element ①


$$N_1 = 3\left(\frac{1}{3} - x\right) \qquad N_2 = 3x$$

$$\left.\frac{\underline{\underline{M}}^e}{\Delta t}\right|_{12} = \int_0^{1/3} 9\left(\frac{x}{3} - x^2\right) = \frac{1}{18\Delta t} = \left.\frac{\underline{\underline{M}}^e}{\Delta t}\right|_{21}$$

$$\left.\frac{\underline{\underline{M}}^e}{\Delta t}\right|_{22} = \int_0^{1/3} 9x^2 = \frac{1}{9\Delta t}$$

$$\left.\frac{\underline{\underline{M}}^e}{\Delta t}\right|_{11} = \int_0^{1/3} 9\left(\frac{1}{3} - x\right)^2 = \frac{1}{18\Delta t}$$

$$\underline{\underline{M}}^e = \begin{bmatrix} +1/9 & 1/18 \\ 1/18 & +1/9 \end{bmatrix} \frac{1}{\Delta t}$$

$$\underline{\underline{K}}^e\Big|_{11} = \int_0^{1/3} -3 \cdot -3 = 3$$

$$\underline{\underline{K}}^e\Big|_{12} = \underline{\underline{K}}\Big|_{21} = \int_0^{1/3} -3 \cdot 3 = -3 \qquad \underline{\underline{K}}^e = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\underline{\underline{K}}^e\Big|_{22} = \int 3 \cdot 3 = 3$$

$$\underline{\underline{C}}^e\Big|_{12} = \int_0^{1/3} 3\left(\tfrac{1}{3}-x\right) 3 = \frac{1}{2}$$

$$\underline{\underline{C}}^e_{21}\Big|_{21} = \int_0^{1/3} 3x\,(-3) = -\frac{1}{2} \qquad \underline{\underline{C}}^e = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\underline{\underline{C}}^e\Big|_{11} = \int_0^{1/3} 3\left(\tfrac{1}{3}-x\right)\cdot(-3) = -\frac{1}{2}$$

$$\underline{\underline{C}}^e\Big|_{22} = \int_0^{1/3} 3x\,(3) = \frac{1}{2}$$

Assembling and substituting into the equation

$$\left[ \frac{1}{\Delta t}\begin{bmatrix} 1/9 & 1/18 & 0 & 0 \\ 1/18 & 2/9 & 1/18 & 0 \\ 0 & 1/18 & 2/9 & 1/18 \\ 0 & 0 & 1/18 & 1/9 \end{bmatrix} + \begin{bmatrix} 5/2 & -5/2 & 0 & 0 \\ -7/2 & 12/2 & -5/2 & 0 \\ 0 & -7/2 & 12/2 & -5/2 \\ 0 & 0 & -7/2 & 7/2 \end{bmatrix} \right] \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}^{n+1}$$

$$= \begin{bmatrix} 1/6 \\ 2/6 \\ 2/6 \\ 1/6 \end{bmatrix} + \frac{1}{\Delta t}\begin{bmatrix} 1/9 & 1/18 & 0 & 0 \\ 1/18 & 2/9 & 1/18 & 0 \\ 0 & 1/18 & 2/9 & 1/18 \\ 0 & 0 & 1/18 & 1/9 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}^{n}$$

Eliminating rows and columns corresponding to $u_0$ & $u_3$

$$\begin{bmatrix} \left(6+\frac{2}{9\Delta t}\right) & \left(\frac{1}{18\Delta t}-\frac{5}{2}\right) \\ \left(\frac{1}{18\Delta t}-\frac{7}{2}\right) & \left(6+\frac{2}{9\Delta t}\right) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^{n+1} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} + \begin{bmatrix} 2/9 & 1/18 \\ 1/18 & 2/9 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^{n}$$

This the expression for solution $u^{n+1}$ with the knowledge of $u^n$ & $\Delta t$

We shall use MATLAB to calculate the solution for $\Delta t = 1$, $\Delta t = 0.5$ & $\Delta t = 0.25$

For the first time step $\Delta t = 1$, $u^n = 0$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 6 + \frac{2}{9} & \frac{1}{18} - \frac{5}{2} \\ \frac{1}{18} - \frac{7}{2} & 6 + \frac{2}{9} \end{bmatrix}^{-1} \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0.0954 \\ 0.1064 \end{bmatrix}$$

$\boxed{2}$ Operator splitting

Steps :

We know $u^n = u_a^n = u_\nu^n$

(1) $\partial_t u_a^{n+1} + \mathcal{L}_a u_a^{n+1} = 0 \longrightarrow$ solve for $u_a^{n+1}$

$$M\left(\frac{U_a^{n+1} - U_a^n}{\Delta t}\right) + C\, U_a^{n+1} = 0$$

$$\left(\frac{M}{\Delta t} + C\right) U_a^{n+1} = \left(\frac{M}{\Delta t}\right) U_a^n \longrightarrow \text{solve for } U_a^{n+1}$$

(2) $u_\nu^n = u_a^{n+1}$

$\partial_t u_\nu^{n+1} + \mathcal{L}_\nu u_\nu^{n+1} = f \longrightarrow$ Solve for $u_\nu^{n+1}$

$$M\left(\frac{U_\nu^{n+1} - U_\nu^n}{\Delta t}\right) + K\, U_\nu^{n+1} = F^{n+1}$$

$$\left(\frac{M}{\Delta t} + K\right) U_\nu^{n+1} = F^{n+1} + \left(\frac{M}{\Delta t}\right) U_\nu^n \longrightarrow \text{solve for } U_\nu^{n+1}$$

③ $\underset{\sim}{U}^{n+1} = \underset{\sim}{U}_{2}^{n+1}$

We solve this again using MATLAB with the known values of matrices $\underset{=}{M}$, $\underset{=}{C}$, $\underset{-}{F}$ and $\underset{=}{K}$

$\boxed{3}$ Now we present the result of MATLAB computation and the error analysis.

## MATLAB code

```
% Task 5

clear all; close all; clc;

j=1;
for value=[0.25 0.5 1]
dt(j) = value;

M = (1/dt(j))*[1/9 1/18 0 0;
          1/18 2/9 1/18 0;
          0 1/18 2/9 1/18;
          0 0 1/18 1/9];

K = [3 -3 0 0;
     -3 6 -3 0;
     0 -3 6 -3;
     0 0 -3 3];

C = [-0.5 0.5 0 0;
     -0.5 0 0.5 0;
     0 -0.5 0 0.5;
     0 0 -0.5 0.5];

F = [1/6;
      1/3;
      1/3;
      1/6];

% Operator splitting solution

nUa = zeros(size(F));
nPlusUa = zeros(size(F));
nPlusUk = zeros(size(F));
nUk = zeros(size(F));
Uprev = zeros(size(F));

for i=1:1/dt(j)
     nPlusUa(2:3,1) = (M(2:3,2:3) + C(2:3,2:3))\(M(2:3,2:3)*Uprev(2:3,1));
     nUk = nPlusUa;
     nPlusUk(2:3,1) = (M(2:3,2:3) + K(2:3,2:3))\( F(2:3,1) + (M(2:3,2:3)*nUk
    (2:3,1)) );
     Uop = nPlusUk;
     Uprev = Uop;
end
Uoper(:,j) = Uop;
fprintf('Operator splitting solution for dt = %f',dt(j));
display(Uop');
j = j+1;
end
```

```

%  Monolithic solution
Umon = zeros(size(F));
Uprev = zeros(size(F));

% for i=1:1/dt(j)
    Umon(2:3,1) = (M(2:3,2:3) + K(2:3,2:3) + C(2:3,2:3))\( F(2:3,1) + (M
    (2:3,2:3)*Uprev(2:3,1)) );
    Uprev = Umon;
% ensd

fprintf('Monolithic solution');
display(Umon');

% Error
errx = [0.25 0.5 1];
for i = 1:3
    err(:,i) = abs(Uoper(:,i) - Umon);
end

plot(errx,err(2,:),'x-b');
hold on;
plot(errx,err(3,:),'*-r');
```

## 3 Error analysis

We evaluate the error in the operator splitting solution for different time step size of 0.25, 0.5 and 1.0 with respect to the monolithic solution. Since only two interior nodes are present, we plot the error at each of the nodes as shown in fig 1. We observe that the error decreases with the decrease in time step size. This shows that with the convenience of opertator splitting, we have to perform computations with smaller time steps.

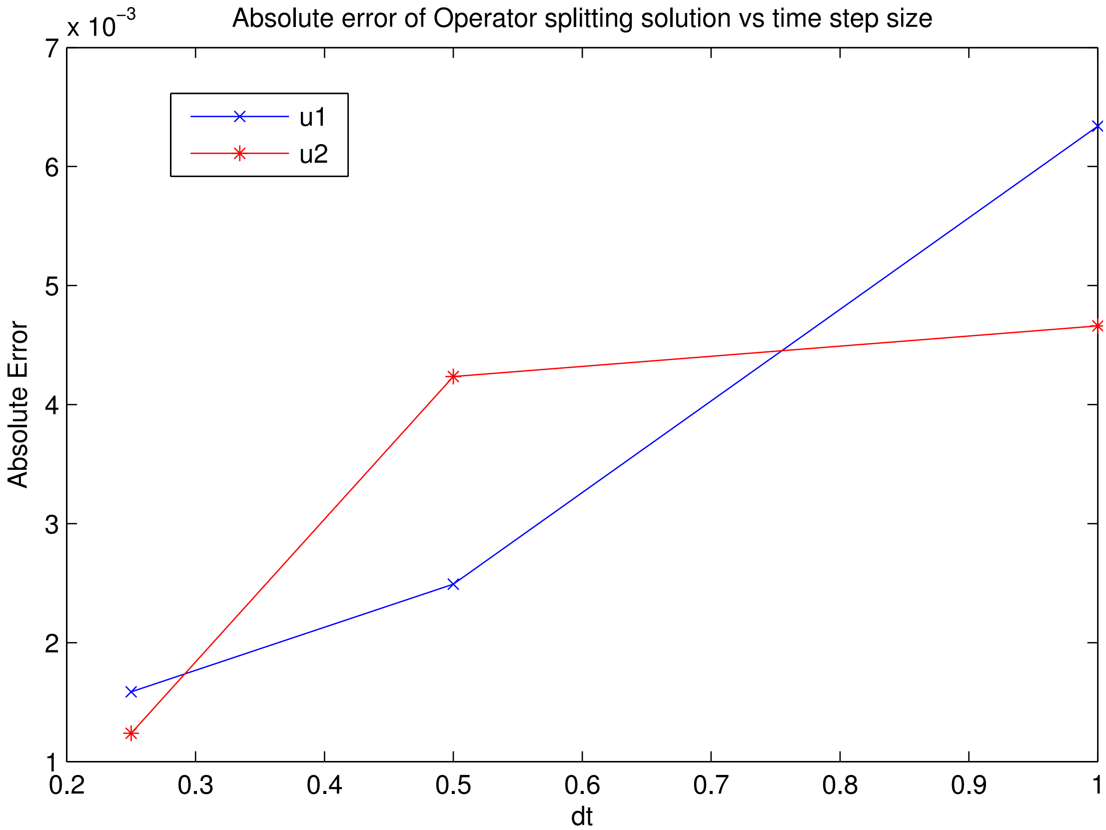



Figure 1: Error of Operator splitting technique with respect to the Monolithic scheme

# COUPLED PROBLEMS

## TASK – 6

## FRACTIONAL STEP METHODS

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$\boxed{1}$ Given

① – $\frac{M}{\Delta t}(\hat{U}^{n+1} - U^n) + K\hat{U}^{n+1} = f - G\hat{p}^{n+1}$

③ – $DM^{-1}Gp^{n+1} = \frac{1}{\Delta t} D\hat{U}^{n+1} - DM^{-1}G\tilde{p}^{n+1}$

② – $\frac{M}{\Delta t}(U^{n+1} - \hat{U}^{n+1}) + \alpha K(U^{n+1} - \hat{U}^{n+1}) + G(p^{n+1} - \tilde{p}^{n+1}) = 0$

What is the optimal value of $\alpha$

Adding ① & ② we should recover the momentum equation

$$\frac{M}{\Delta t}(\hat{U}^{n+1} - U^n + U^{n+1} - \hat{U}^{n+1})$$

$$+ K(\hat{U}^{n+1} + \alpha U^{n+1} - \alpha\hat{U}^{n+1}) \qquad = f$$

$$+ G(p^{n+1} - \check{p}^{n+1} + \check{p}^{n+1})$$

Only when we substitute $\alpha = 1$, we recover the momentum equation

$$\frac{M}{\Delta t}(U^{n+1} - U^n) + KU^{n+1} + Gp^{n+1} = f$$

Hence $\alpha = 1$. This is also verified in the reference (Quarteroni 2000) where Yosida scheme was first introduced. In particular, the incremental formulation shown in Remark 6.1 (pg 519) confirms this.

## 2 Source of error in the scheme.

Yoshida scheme is a scheme introduced to solve unsteady incompressible N-S equation. It belongs to a class of methods based on splitting the original problem into the successive solution of smaller problems, involving velocity and pressure fields separately. this is based on the block factorization of matrices obtained after tim-space discretizations of the original problem.

Yoshida scheme is regarded as a "quasi-compressibility" scheme since the approximation induced by the splitting affects the continuity equation. this is in contrast to the so-called "projection methods"

In Yoshida scheme a 'small' perturbation to continuity equation is added to stabilize the solution. this is because, such an addition, if done suitably, can be used to circumvent the LBB condition. This way linear FE can be adopted for bothe velocity and pressure approximations without spurious pressure nodes.

the perturbation may be of the following form

① $\nabla \cdot \underline{u} + \epsilon \frac{\partial p}{\partial t} = 0$ , $P(t=0) = P_o$

artificial compressibility method

② $\nabla \cdot \underline{u} + \epsilon p = 0$

penalty method

③ $\nabla \cdot \underline{u} + \epsilon \Delta p = 0$ , $\nabla p \cdot \underline{n}|_{\Gamma} = 0$

Petrov - Galerkin method

these are explained in reference (Quarteroni 1999)

$\epsilon$ must be large enough to have a significant regularizing effect but small enough to minimize perturbations.

Hence added term for regularization with the effect of stabilizing the whole system, affects the continuity equation. Hence Yosida scheme guarantees continuity of momentum but not the conservation of mass.

Hence the main source of the error for Yosida scheme is the unsatisfied continuity equation. This error can be reduced to within the tolerance limits as shown in reference (Quarteroni 2000)

References:


(1) Quarteroni 1991
Analysis of the Yoshida method for the incompressible Navier - Stokes equation
J. Math. Pures. Appl 78, 1999 pg 473-503

(2) Quarteroni 2000
Factorization methods for the numerical solution of approximation of Navier - Stokes equation
Comput. Methods Appl. Mech. Engrg.
188, 2000 pg 505 - 526

# COUPLED PROBLEMS

## TASK - 7

## ALE FORMULATION

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$\boxed{1}$ Given $r(x, y, z, t) = [2x, ye^t, z]$

Eqn of movement

$$x = Xe^t$$
$$y = Y + e^t - 1 \qquad \text{—— (1)}$$
$$z = Z$$

Eqn of Movement of mesh

$$x_m = \mathcal{X} + \alpha t$$
$$y_m = \mathcal{Y} - \beta t$$
$$z_m = \mathcal{Z}$$

(a) Description of $r(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$

Using (1)

$$r(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = [2(\mathcal{X} + \alpha t), e^t(\mathcal{Y} - \beta t), \mathcal{Z}]$$

(b) velocity of particles $v = \frac{\partial}{\partial t} x(X, Y, Z)$

$$= \frac{\partial}{\partial t} \varphi(X, Y, Z) = [e^t X, e^t, 0]$$

velocity of mesh $= \frac{\partial}{\partial t} x(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = \frac{\partial}{\partial t} \Phi(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$

$$v_{MESH} = [\alpha, -\beta, 0]$$

(c) $\frac{\partial}{\partial t} r_{ALE}(\mathcal{X}(X, t), t)$

$$= \frac{\partial}{\partial t} r_{ALE}(\mathcal{X}, t) + (v - v_{mesh}) \cdot \nabla r(x, t)$$

$$= [2\alpha, \mathcal{Y}e^t - \beta t e^t - \beta e^t, 0] + (v - v_{mesh}) \cdot \nabla r(x, t)$$

$$= \begin{bmatrix} 2\alpha \\ Ye^t - \beta te^t - \beta e^t \\ 0 \end{bmatrix}^T + \begin{bmatrix} e^t X - \alpha \\ e^t + \beta \\ 0 \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cancel{\alpha} + e^t X \cdot 2 - \cancel{2\alpha} \\ Ye^t - \beta te^t - \cancel{\beta e^t} + e^{2t} + \cancel{\beta e^t} \\ 0 \end{bmatrix}^T$$

$$= \left[ 2Xe^t \,,\, -\beta te^t + e^{2t} + Ye^t \,,\, 0 \right]$$

$\boxed{2}$ ALE form of incompressible N-S equation

To get the AEE form of N-S equation from the Eulerian form, we need to replace the convective terms, the material velocity $u(x,t)$ with the convective velocity $c = u - u_{mesh}$

Mass:

$$\underbrace{\frac{\partial \rho(x,t)}{\partial t}}_{\text{evaluated at mesh points}} + \underbrace{\underline{c} \cdot \underline{\nabla} \rho = -\rho \nabla \cdot \underline{u}}_{\text{evaluated in the spacial coordinate}}$$

With incompressibility : $\nabla \cdot \underline{u} = 0$

Momentum:

$$\rho \left[ \underbrace{\frac{\partial \underline{v}(x,t)}{\partial t}}_{\text{evaluated at mesh points}} + \underbrace{(\underline{c} \cdot \nabla) \underline{v} \Big] = \nabla \cdot \underline{\underline{\sigma}} + \rho \underline{b}}_{\text{evaluated in the spacial coordinates.}} \right.$$

With incompressibility, $\rho = \text{constant}$ and

with Newtonian fluids $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \mu \nabla_s \underline{u}$

$$\therefore \frac{\partial \underline{v}(x,t)}{\partial t} + (\underline{c}\cdot\nabla)\underline{v} = -\cancel{p}\frac{\nabla p}{\rho} + [\Delta \underline{u}]\nu + \underline{b}$$

When discretized in time, temporal derivatives are computed as the difference between values of the property at the moving nodes.

# $\boxed{3}$ Bibliography on mesh movement in ALE

Mesh update strategies can be either of the two:

1. Mesh regularization: It is to keep the computational mesh as regular as possible and to avoid mesh entanglement. Mesh regularization requires that updated nodal coordinates be specified at each station of a calculation, either through step displacements, or from current mesh velocities. It can be further classified as:

   (a) When the motion of the material surfaces (usually the boundaries) is known a priori, the mesh motion is also prescribed a priori. In general, this implies a Lagrangian description at the moving boundaries, while a Eulerian formulation is employed far away from the moving boundaries. Papers on these techniques Huerta and Liu (1988a) and Rodriguez-Ferran et al. (2002).

   (b) In all other cases, at least a part of the boundary is a material surface whose position must be tracked at each time step. Thus, a Lagrangian description is prescribed along this surface (or at least along its normal). Papers with this technique are Noh (1964) and Liu and Chang (1984). In fluid–structure interaction problems, solid nodes are usually treated as Lagrangian, while fluid nodes are treated as fixed or updated according to some simple interpolation scheme.

   Some other techniques include

   (a) Transfinite mapping method: was originally designed for creating a mesh on a geometric region with specified boundaries. It induces a very low-cost procedure, since new nodal coordinates can be obtained explicitly once the boundaries of the computational domain have been discretized. Papers describing these are Ponthot and Hogge (1991), Yamada and Kikuchi (1993)

   (b) Laplacian Smoothing and Variational Methods: This involves solving the laplacian or poisson problem to rezone the nodes describing the mesh so that a smooth distribution of them are obtained. Examples include Liu et al. (1988), Ghosh and Kikuchi (1991)

   (c) Mesh smoothing and simple interpolations: The goal of this method is to minimize both the squeeze and distortion of each element in the mesh. It uses mesh-smoothing algorithm designed to improve the shape of the elements once the topology is fixed. Examples include Donea et al. (1982), Batina (1991)

2. Mesh adaptation: It is to concentrate elements in zones of steep solution gradient. This is basically used as a mesh refinement technique. Some papers on this include Huerta et al. (1999) and Askes and Rodriguez-Ferran (2001)

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# Coupled Problems
# Fluid-Structure Interaction

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## 1 Added Mass Effect

Added mass effect occurs in FSI problems when the partitioned schemes do not converge to a solution. This is due to the similarity in densities of the solid and fluid phases such as arterial flow, biomechanics, etc,. The added mass operator describes, how the prediction of the interface acceleration relates to the new interface forces for the structure problem. The added mass operator acts as additional mass on the degrees of freedom on the interface. It can be mitigated by techniques such as Aitken's relaxation scheme, steepest-descent method and using robin-robin boundary conditions

## 2 Aitken Relaxation

The code snipped used to apply the relaxation scheme is given below and the plot of the problem solution is shown in fig 1.

```
close all
clear variables

count = 1;
w = 1; %initial w
HeatProblem.Solution.uRight = 0;
HeatProblem1.Solution.uLeft = 0;
ug1 = 0; ug2 = 0;

%Problem1
%Domain
Data.inix = 0;
Data.endx = 0.4;
Data.nelem = 2;
%Physical
Data.kappa = 1;
Data.source = 1;
%Boundary conditions
%Dirichlet
Data.FixLeft = 1; %0, do not fix it , 1: fix it
Data.LeftValue = 0;
Data.FixRight =0;
Data.RightValue = 0;
%Neumann
Data.FixFluxesLeft = 0;
Data.LeftFluxes = 0;
Data.FixFluxesRight = 1;
```

```
Data.RightFluxes = 0;

%Problem2
%Domain
Data1.inix = 0.4;
Data1.endx = 1;
Data1.nelem = 3;
%Physical
Data1.kappa = 1;
Data1.source = 1;
%Boundary conditions
%Dirichlet
Data1.FixLeft = 1; %0, do not fix it, 1: fix it
Data1.LeftValue = 0;
Data1.FixRight =1;
Data1.RightValue = 0;
%Neumann
Data1.FixFluxesLeft = 0;
Data1.LeftFluxes = 0;
Data1.FixFluxesRight = 0;
Data1.RightFluxes = 25;



itercounter = 1; errper = 100; usolold = 0; i1 = 1;

while (errper > 1e-6 && itercounter < 100)

     dom1before = HeatProblem.Solution.uRight;

%Problem 1
HeatProblem = HP_Initialize(Data);
HeatProblem = HP_Build(HeatProblem);
HeatProblem = HP_Solve(HeatProblem);

    dom1after = HeatProblem.Solution.uRight;

    %Aitkin Relaxation scheme
    if (i1>2)
       w = (ug2 - ug1) / (ug2 - ug1 + HeatProblem.Solution.uRight - usolold)
    ;
    end
    wcount(itercounter,1) = w;

    Data1.LeftValue = ug1 + w*(HeatProblem.Solution.uRight - ug1);
    ug2 = ug1;
    ug1 = Data1.LeftValue;

    dom2before = HeatProblem1.Solution.uLeft;

%Problem 2
```

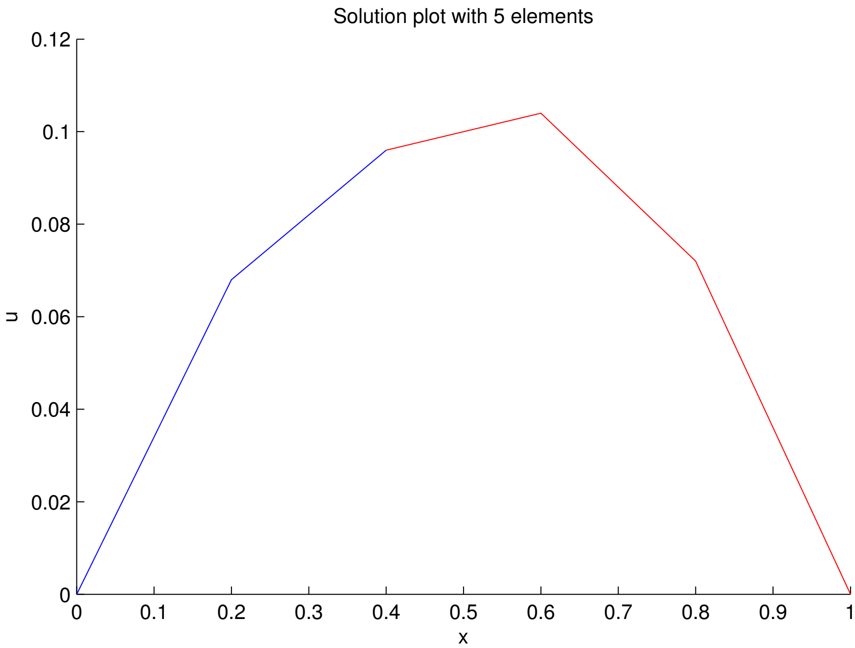



Figure 1: Solution with Aitken's relaxation scheme

```
HeatProblem1 = HP_Initialize(Data1);
HeatProblem1 = HP_Build(HeatProblem1);
HeatProblem1 = HP_Solve(HeatProblem1);

    dom2after = HeatProblem1.Solution.uLeft;

    %Update for data
    Data.RightFluxes = -HeatProblem1.Solution.FluxesLeft;

errper = abs(abs(HeatProblem.Solution.uRight - usolold)*100/usolold);
% fprintf('%f\n',errper);


errdom1(itercounter,1) = abs(dom1before - dom1after);
errdom2(itercounter,1) = abs(dom2before - dom2after);

itercounter = itercounter +1;
usolold = HeatProblem.Solution.uRight;
i1 = i1+1;
end

% Solve and plot
HP_Plot(HeatProblem,1,1);
HP_Plot(HeatProblem1,1,2);
```

3. The monolithic treatment of the transient diffusion equation with ~~3~~ elements $(h = 0.25)$ leads to a $4\times4$ matrix linear system of equation as follows

$$\left(\frac{M}{\Delta t} + K\right) U^{n+1} = F + \frac{M}{\Delta t} U^n$$

Without the b.c at the boundaries, the system is $4\times4$

$M \rightarrow$ mass matrix

$K \rightarrow$ stiffness matrix

For simplicity let it be written as

$$Ax = b$$

where

See pg 8-5 for correction

$$A = \begin{bmatrix} A_{11} & \cdots & & A_{14} \\ A_{21} & A_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{41} & \cdots & & A_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

The boundary conditions are $x_1 = 0$ and $x_4 = 0$
We can write it in the matrix form as

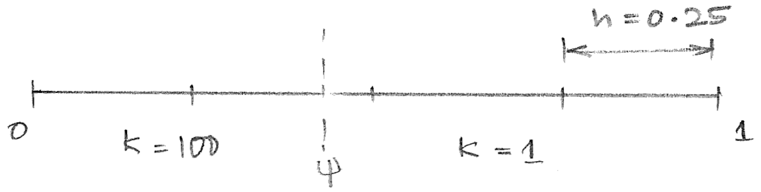
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow Bx = h$$

Lagrange multipliers method of prescribing boundary condition when written in discrete form is given by

$$\left[\begin{array}{c|c} A & B \\ \hline B & 0 \end{array}\right] \left[\begin{array}{c} x \\ \hline \lambda \end{array}\right] = \left[\begin{array}{c} b \\ \hline h \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} A_{11} & A_{12} & A_{13} & A_{14} & 1 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} & 0 & 0 & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} & 0 & 0 & 0 & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \hline \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{array}\right] = \left[\begin{array}{c} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

$\boxed{4}$



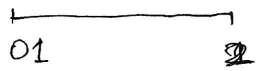

We shall build the system of matrices for this problem
The discrete equation for transient diffusion equation are

$$\left( \frac{M}{\Delta t} + K \right) U^{n+1} = F - \frac{M}{\Delta t} U^n$$

As previously calculated, the terms above need to be revised. But diffusion coefficient affects only the K matrix and hence we shall see how this is calculated.

$$K_{ij} = \int_{\Omega} k \frac{dN_i}{dx} \frac{dN_j}{dx} \, d\Omega$$

As calculated for problem 5, in element ①




$$N_1 = \left(\frac{1}{4} - x\right) 4 \quad , \quad N_2 = 4x$$

$$K^e_{ij} = \int_{\Omega_e} k^e \frac{dN_i}{dx} \frac{dN_j}{dx}$$

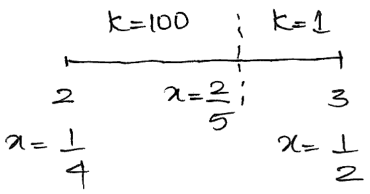
Since the term 'k' (diffusion coefficient) is constant in the element, it can be treated as a constant and the computation is simpler

$$K^e_{11} = k^e \int_0^{1/4} (-4)(4) = 4 k_e = 400 = K^e_{22}$$

$$K^e_{12} = k_e \int_0^{1/4} (-4)(4) = -4 k_e = -400$$

$$\therefore \quad K^e = \begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix}$$

For element ②




$$N_1 = -4x + 2$$

$$N_2 = 4x - 1$$

$$k_{11}^{e} = 100 \int_{1/4}^{2/5} (4)\cdot(4) + 1\int_{2/5}^{1/2} (4)\cdot(4)$$

$$= 100\,(4)(4)(0.15) + 1\,(4)(4)(0.1)$$

$$= 240 + 2.4$$

$$= 242.4 = k_{22}^{e}$$

$$k_{12}^{e} = 100 \int_{1/4}^{2/5} (-4)\,(4) + 1\int_{2/5}^{1/2} (4)\,(4)$$

$$= -242.4$$

$$\therefore\ k^{e} = \begin{bmatrix} 242.4 & -242.4 \\ -242.4 & 242.4 \end{bmatrix}$$

In element 3 & 4 with $k = 1$

$$k^{e} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

$\therefore$ Assembling $k$

$$k = \begin{bmatrix} 400 & -400 & 0 & 0 & 0 \\ -400 & 642.4 & -242.4 & 0 & 0 \\ 0 & -242.4 & 246.4 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix}$$

The other matrix is the mass matrix given in each element as.

$$M^{e} = \frac{1}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The calculation is similar to the problem 5

Assembling

$$M = \frac{1}{12}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \qquad F = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/2 \\ 1/4 \end{bmatrix}$$

Thus the discrete system is completly calculated.

$\boxed{3}$ Comment on condition number

In this problem, if $\Delta t = 1$, $A = K + M$

$$K = \begin{bmatrix} 4 & -4 & 0 & 0 & 0 \\ -4 & 8 & -4 & 0 & 0 \\ 0 & -4 & 8 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix} \qquad M = \frac{1}{12}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Condition number of $A = 36.8669$

Condition number with lagrange multipliers $= \hat{A} = \left[\begin{array}{c|c} A & B \\ \hline B & 0 \end{array}\right] =$

<u>Correction</u>

Since we assumed 3 elements, it is incorrect. We perform the same with 4 elements. In that case the sizes of matrices change

$Ax = b$ becomes

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{15} \\ A_{21} & A_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{51} & \cdots & \cdots & A_{55} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_5 \end{bmatrix}$$

$Bx = h$ becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we assemble the system for applying lagrange multipliers

$$\left[\begin{array}{c|c} A & B \\ \hline B & 0 \end{array}\right] \left[\begin{array}{c} x \\ \hline \lambda \end{array}\right] = \left[\begin{array}{c} b \\ \hline h \end{array}\right]$$

The condition number of this expanded system
$\Rightarrow$ very high

Hence using lagrange multipliers increases the condition number which makes it harder to solve the linear system. The convenience of applying dirichlet condition via lagrange multipliers comes at a cost!