[Coupled Problems Home works
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Exercise 1: TRANSMISSION CONDITIONS

Case 1: EULER - BERNOULLI BEAM

equation for the vertical deflection : $EI \dfrac{d^4 v}{dx^4} = f$

B.C

$$v(0) = 0 \quad , v(L) = 0 \quad , \theta(0) = v'(0) = 0 \quad , \theta(L) = v'(L) = 0$$

Principle of Virtual Work

$$EI \int_0^L \frac{d^2 \delta v}{dx^2} \frac{d^2 v}{dx^2} dx = \int_0^L \delta v \, f \, dx$$

for all virtual δv such that: $\begin{cases} \delta v(0) = 0 \\ \delta v(L) = 0 \\ \dfrac{d\delta v}{dx}(0) = 0 \\ \dfrac{d\delta v}{dx}(L) = 0 \end{cases}$

Question (a)

Given the weak form of the problem reported above
we conclude that the following inequality must be satisfied

$$EI \int_0^L \frac{d^2 \delta v}{dx^2} \frac{d^2 v}{dx^2} dx < \infty \quad \text{and} \quad \int_0^L \delta v \, f < \infty$$

this means already that $\delta v, v \in L^2(\Omega)$ and $\dfrac{d\delta v}{dx}, \dfrac{dv}{dx} \in L^2(\Omega)$

More over, we can use the following expression:

$$\int_0^L \left(\frac{d\delta v}{dx}\right)^2 dx = \cancel{\delta v \frac{d\delta v}{dx}}\Bigg|_{\substack{x=0 \\ x=L}} - \int_0^L \delta v \frac{d^2 \delta v}{dx^2} dx = - \int_0^L \delta v \frac{d^2 \delta v}{dx^2} dx$$

$\to 0$ (given the B.C.)

Since the right hand side term is bounded $\left(\int_0^L \left(\frac{d^2\delta v}{dx^2}\right)^2 dx < \infty\right)$ then the first term (LHS) must be bounded too. $\left(\int_0^L \left(\frac{d\delta v}{dx}\right)^2 dx < \infty\right)$

so $\frac{d\delta v}{dx}, \frac{dv}{dx} \in L^2(\Omega)$.

$\Rightarrow v, \delta v \in H^2(\Omega)$, where

$$H^2(\Omega) = \left\{ v \in \Omega \to \mathbb{R} : \int_\Omega v^2 < \infty, \int_\Omega \left(\frac{dv}{dx}\right)^2 < \infty, \int_\Omega \left(\frac{d^2 v}{dx^2}\right)^2 < \infty \right\}$$

and $\Omega = [0, L]$

but, given the boundary conditions $\left(\begin{array}{l} v = 0 \text{ on } \partial\Omega, \ v' = 0 \text{ on } \partial\Omega \\ \text{or } \delta v = 0 \text{ on } \partial\Omega, \ \delta v' = 0 \text{ on } \partial\Omega \end{array}\right)$

Where $\partial\Omega = \{(0); (L)\}$

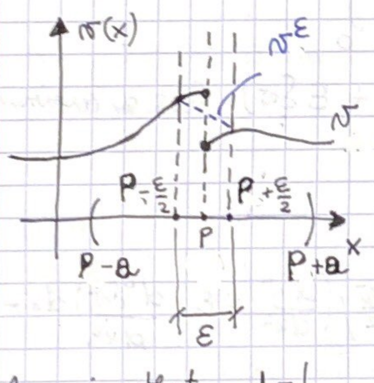
then $v, \delta v \in H^2_0(\Omega) = \{ v \in H^2(\Omega) : v = 0, v' = 0 \text{ on } \partial\Omega \}$

Question (b)

Given $[0, L] = [0, P] \cup (P, L]$ let's obtain the transmission conditions given by the regularities in P.

If v ~~is~~ discontinuous in P, they cannot be in $H^2(\Omega)$
and $\frac{dv}{dx}$

- Let's prove that: first, let's focus on "v"



v^ε: regularized deflection

so that: $v = \lim_{\varepsilon \to 0} v^\varepsilon$

Assuming that $\left.\frac{dv}{dx}\right|_P = \lim_{\varepsilon \to 0} \left.\frac{d v^\varepsilon}{dx}\right|_P$

doing
$$\int_{P-a}^{P+a} \frac{dv^\varepsilon}{dx} dx = \int_{P-a}^{P-\varepsilon/2} \frac{dv}{dx} dx + \int_{P-\varepsilon/2}^{P+\varepsilon/2} \frac{dv^\varepsilon}{dx} dx + \int_{P+\varepsilon/2}^{P+a} \frac{dv}{dx} dx$$

which is equal to: $\displaystyle\int_{p-a}^{p-\varepsilon/2} \frac{dv}{dx}\,dx + \cancel{\varepsilon}\left[\frac{v(p+\varepsilon/2) - v(p-\varepsilon/2)}{\cancel{\varepsilon}}\right] + \int_{p+\frac{\varepsilon}{2}}^{p+a} \frac{dv}{dx}\,dx$

doing the limit: $\displaystyle\lim_{\varepsilon\to\emptyset} = \int_{p-a}^{p} \frac{dv}{dx}\,dx + \underbrace{v(p^+) - v(p^-)}_{*} + \int_{p}^{p+a} \frac{dv}{dx}\,dx$

the whole integral of the first derivative of a discontinuous function is correct and makes sense.

We call $*$ the jump of v in P : $[\![v]\!]_p = v(p^+) - v(p^-)$

Using the Heaviside function and its derivative (first) Delta di Dirac :

$$H(x) = \begin{cases} 0 & \text{for } x<0 \\ 1 & \text{for } x\geq 0 \end{cases} \quad ; \quad \delta(x) = \begin{cases} 0 & \text{if } x\neq 0 \\ \infty & \text{if } x=0 \end{cases}$$

Knowing that $\displaystyle\int_{-a}^{a} \frac{dH}{dx}\,dx = \int_{-a}^{a} \delta(x)\,dx = H(a) - H(-a) = 1$

becomes $\displaystyle\int_{-a}^{a} f\,\frac{dH}{dx}\,dx = \int_{-a}^{a}\left[\frac{d}{dx}(fH) - H\frac{df}{dx}\right]dx = f(a) - \int_{0}^{a}\frac{df}{dx}\,dx = f(0)$

For this case :

$$\int_{P-a}^{P+a}\left(\frac{dv^{\varepsilon}}{dx}\right)^2 dx = \int_{P-a}^{P-\varepsilon/2}\left(\frac{dv}{dx}\right)^2 dx + \cancel{\varepsilon}\left(\frac{(v(p+\varepsilon/2) - v(P-\varepsilon/2))^2}{\varepsilon^{\cancel{2}}}\right) + \int_{P+\varepsilon/2}^{P+a}\left(\frac{dv}{dx}\right)^2 dx$$

doing $\displaystyle\lim_{\varepsilon\to 0}\int_{P-a}^{P+a}\left(\frac{dv^{\varepsilon}}{dx}\right)^2 dx = \infty$ which contrasts with the definition of the space of functions for the v, and behaves like the Delta δ in $\emptyset$ ($\uparrow+\infty$!) $\rightarrow \displaystyle\int_{-a}^{a}\delta^2\,dx = \infty$

$\Rightarrow$ the function v must be continous !

in other words: $\boxed{[\![v]\!]_p = v(p^+) - v(p^-) = \emptyset \;!}$

(first trasmission condition)

- Focusing on the first derivative of v : $\dfrac{dv}{dx}$

Using the same picture reported above, we define :

$\dfrac{dv^{\varepsilon}}{dx}$: regularized function for the first derivative of v

$$\frac{dv}{dx} = \lim_{\varepsilon\to 0} \frac{dv^{\varepsilon}}{dx} \quad ; \text{ assuming that } \left.\frac{d^2v}{dx^2}\right|_p = \lim_{\varepsilon\to 0} \left.\frac{dv^{\varepsilon}}{dx}\right|_p$$

$$\int_{p-a}^{p+a} \frac{d^2v^{\varepsilon}}{dx^2}dx = \int_{p-a}^{p-\varepsilon/2} \frac{d^2v}{dx^2}dx + \int_{p-\varepsilon/2}^{p+\varepsilon/2} \frac{d^2v^{\varepsilon}}{dx^2}dx + \int_{p+\varepsilon/2}^{p+a} \frac{d^2v}{dx^2}dx =$$

$$= \int_{p-a}^{p-\varepsilon/2} \frac{d^2v}{dx^2}dx + \cancel{\varepsilon}\left[\frac{\frac{dv}{dx}(p+\varepsilon/2) - \frac{dv}{dx}(p-\varepsilon/2)}{\cancel{\varepsilon}}\right] + \int_{p+\varepsilon/2}^{p+a} \frac{d^2v}{dx^2}dx$$

doing the limit $\displaystyle\lim_{\varepsilon\to 0}\int_{p-a}^{p+a} \frac{d^2v^{\varepsilon}}{dx^2}dx = \int_{p-a}^{p} \frac{d^2v}{dx^2}dx + \underbrace{\left[\frac{dv}{dx}(p^+) - \frac{dv}{dx}(p^-)\right]}_{*} +$

$+ \displaystyle\int_{p}^{p+a} \frac{d^2v}{dx^2}dx$, it's possible to see that the integral of the first derivative of a discontinous function makes sense

The jump of $\dfrac{dv}{dx}$ $*$ around p is defined as: $\left[\!\left[\dfrac{dv}{dx}\right]\!\right]_p = \left[\dfrac{dv}{dx}(p^+) - \dfrac{dv}{dx}(p^-)\right]$.

But, doing the integral of the second derivative:

$$\int_{p-a}^{p+a}\left[\frac{d}{dx}\left(\frac{dv^{\varepsilon}}{dx}\right)\right]^2 dx = \int_{p-a}^{p-\varepsilon/2}\left(\frac{d^2v}{dx^2}\right)^2 dx + \cancel{\varepsilon}\left(\frac{\left(\frac{dv}{dx}(p+\varepsilon/2) - \frac{dv}{dx}(p-\varepsilon/2)\right)^2}{\varepsilon^{\cancel{2}}}\right) + \int_{p+\varepsilon/2}^{p+a}\left(\frac{d^2v}{dx^2}\right)^2 dx$$

doing the limit for $\varepsilon\to 0$, $\displaystyle\lim_{\varepsilon\to 0}\int_{p-a}^{p+a}\left[\frac{d}{dx}\left(\frac{dv^{\varepsilon}}{dx}\right)\right]^2 dx = \infty$ but it contrasts with the definition of the space of functions for the v and $\dfrac{dv}{dx}$, and behaves like, $\displaystyle\int_{-a}^{a}\delta^2\, dx = \infty$

$\Rightarrow$ the function $\dfrac{dv}{dx}$ then must be continous!

In other words : $\boxed{\left[\!\left[\dfrac{dv}{dx}\right]\!\right]_p = \dfrac{dv}{dx}(p^+) - \dfrac{dv}{dx}(p^-) = \varnothing\ !}$

(Second transmission condition)

Question (C)

- let us focus on the unknown "v":

$$\forall \delta v : EI \int_0^L \frac{d^2 \delta v}{dx^2} \frac{d^2 v}{dx^2} dx = \int_0^L \delta v f \, dx$$

with $\delta v(0) = 0$, $\delta v(L) = 0$

let's take the original weak form and focus only on one subdomain:

$$EI \int_0^P \delta v \frac{d^4 v}{dx^4} dx = \int_0^P \delta v f \, dx$$

focusing only on the LHS term:

$$EI \int_0^P \delta v \frac{d^4 v}{dx^4} dx = EI \int_0^P \frac{d}{dx}\left(\delta v \frac{d^3 v}{dx^3}\right) dx - EI \int_0^P \frac{d\delta v}{dx} \cdot \frac{d^3 v}{dx^3} dx$$

$$= \underbrace{EI \, \delta v \cdot \frac{d^3 v_1}{dx^3}\Big|_P}_{\text{Here the divergence theorem has been used; the normal } \underline{n}_1 \text{ has the same direction of x-axis}} - EI \int_0^P \frac{d\delta v}{dx} \cdot \frac{d^3 v}{dx^3} dx$$

the same calculation can be done for the $(P,L]$ subdomain, but with a normal vector that has direction like "$-x$-axis"

Using additivity:

$$\underbrace{EI \int_0^P \delta v \frac{d^4 v}{dx^4} dx + EI \int_P^L \delta v \frac{d^4 v}{dx^4} dx}_{*} = \delta v \cdot EI \frac{d^3 v_1}{dx^3}\Big|_P - EI \int_0^P \frac{d\delta v}{dx} \cdot \frac{d^3 v}{dx^3} dx$$

$$- \delta v \cdot EI \frac{d^3 v_2}{dx^3}\Big|_P - EI \int_P^L \frac{d\delta v}{dx} \frac{d^3 v}{dx^3} dx =$$

$$= \underbrace{- EI \int_0^L \frac{d\delta v}{dx} \frac{d^3 v}{dx^3} dx}_{*} + EI \left(\frac{d^3 v_1}{dx^3} - \frac{d^3 v_2}{dx^3}\right)\Big|_P \delta v \, ,$$ due to additivity and the fact that $[\![\delta v]\!]_P = 0$

Since the integral of * must be equal to *,

$$EI \left(\frac{d^3 v_1}{dx^3} - \frac{d^3 v_2}{dx^3} \right) \Bigg|_P \delta v = 0 \quad \forall \delta v \in H^2(\Omega)$$

and $\delta v = 0$ on $\partial\Omega$ $\quad (\Omega \equiv [0, L])$

$$\downarrow$$

$$\boxed{\begin{aligned} -EI \frac{d^3 v_1}{dx^3} &= -EI \frac{d^3 v_2}{dx^3} \quad \text{on } P \\ Q_1 &= Q_2 \quad \text{on } P \end{aligned}}$$

third transmission condition (equality of shear)

- let us focus on the first derivative of the unknown, which is continous too (II t.c.) : "$\frac{dv}{dx}$":

$$\forall \frac{d\delta v}{dx} : EI \int_0^L \frac{d^2 \delta v}{dx^2} \frac{d^2 v}{dx^2} dx = \int_0^L \delta v \, f \, dx$$

with $\frac{d\delta v}{dx}(0) = 0$, $\frac{d\delta v}{dx}(L) = 0$

let us take the original weak form and focus only on one subdomain:

$$EI \int_0^P \delta v \frac{d^4 v}{dx^4} dx = \int_0^P \delta v \, f \, dx$$

focusing only on the LHS term:

$$EI \int_0^P \delta v \frac{d^4 v}{dx^4} dx = EI \int_0^P \frac{d}{dx} \left(\frac{d\delta v}{dx} \frac{d^2 v}{dx^2} \right) dx - EI \int_0^P \frac{d^2 \delta v}{dx^2} \frac{d^2 v}{dx^2} dx$$

$$= \underbrace{EI \frac{d\delta v}{dx} \cdot \frac{d^2 v_1}{dx^2} \Bigg|_P}_{} - EI \int_0^P \frac{d^2 \delta v}{dx^2} \frac{d^2 v}{dx^2} dx$$

Here the divergence theorem has been used the normal $\underline{n}_1$ has the same direction of x-axis

the same calculation can be done for the other subdomain $[P, L]$, but with a normal vector $\underline{n}_2$ directed like -"x-axis"

Using additivity:

$$EI\int_0^P \delta v\, \frac{d^4 v}{dx^4}\,dx + EI\int_P^L \delta v\,\frac{d^4 v}{dx^4}\,dx = \frac{d\delta v}{dx}\,EI\,\frac{d^2 v_1}{dx^2}\Big|_P - EI\int_0^P \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2}\,dx$$

$$-\frac{d\delta v}{dx}\,EI\,\frac{d^2 v_2}{dx^2}\Big|_P - EI\int_P^L \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2}\,dx =$$

$$= \underbrace{-EI\int_0^L \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2}\,dx}_{*} + EI\left(\frac{d^2 v_1}{dx^2} - \frac{d^2 v_2}{dx^2}\right)\Big|_P \frac{d\delta v}{dx}$$

due to additivity and the fact that $[\![\frac{d\delta v}{dx}]\!]_P = \emptyset$

Since the integral * must be equal to *,

$$EI\left(\frac{d^2 v_1}{dx^2} - \frac{d^2 v_2}{dx^2}\right)\Big|_P \frac{d\delta v}{dx} = 0 \qquad \forall\ \delta v \in H^2(\Omega)$$

or $\frac{d\delta v}{dx} \in H^1(\Omega)$ $\quad (\Omega \equiv [0,L])$

with $\frac{d\delta v}{dx} = 0$ on $\partial\Omega$

$$\boxed{\begin{aligned} EI\,\frac{d^2 v_1}{dx^2} &= EI\,\frac{d^2 v_2}{dx^2} \quad \text{on } P \\ M_1 &= M_2 \quad \text{on } P \end{aligned}}$$

fourth transmission condition (equality of bending moments)

Case 2 : MAXWELL PROBLEM

find a vector $\underline{u} : \Omega \to \mathbb{R}^3$ such that
$$\begin{cases} \nu\,\underline{\nabla}\times\underline{\nabla}\times\underline{u} = \underline{f} & \text{in } \Omega \\ \underline{\nabla}\cdot\underline{u} = 0 & \text{in } \Omega \\ \underline{n}\times\underline{u} = \underline{0} & \text{on } \partial\Omega \end{cases}$$

Where $\nu > 0$ $\underline{f}$ is a div-free force field an $\underline{n}$ the unit external normal

the equation (2) is redundant

Question (a)

Let's use $\delta\underline{u}$ such that $\underline{n}\times\delta\underline{u} = \underline{0}$ on $\partial\Omega$, and:

$$\int_\Omega \delta\underline{u}\cdot(\nu\underline{\nabla}\times\underline{\nabla}\times\underline{u})\,d\Omega = \int_\Omega \delta\underline{u}\cdot\underline{f}\,d\Omega \qquad \forall\,\delta\underline{u}$$

this means: $\displaystyle\int_\Omega \delta\underline{u}\cdot(\nu\,\underline{\nabla}\times\underline{\nabla}\times\underline{u})\,d\Omega = \nu\int_\Omega \underline{\nabla}\cdot\left((\underline{\nabla}\times\underline{u})\times\delta\underline{u}\right)d\Omega\ +$

$$+\int_{\Omega} \nu\, (\underline{\nabla}\times \delta\underline{u})\cdot(\underline{\nabla}\times\underline{u})\, d\Omega = *$$

(this is due to the fact that:

$$\underline{\nabla}\cdot(\underline{a}\times\underline{b}) = (\underline{\nabla}\times\underline{a})\cdot\underline{b} - (\underline{\nabla}\times\underline{b})\cdot\underline{a}\ )$$

where $\underline{a} = \underline{\nabla}\times\underline{u}$

$\underline{b} = \delta\underline{u}$

$$* = \nu\int_{\Omega} \underline{\nabla}\cdot((\underline{\nabla}\times\underline{u})\times\delta\underline{u})\, d\Omega + \nu\int_{\Omega}(\underline{\nabla}\times\delta\underline{u})\cdot(\underline{\nabla}\times\underline{u})\, d\Omega = \int_{\Omega}\underline{f}\cdot\delta\underline{u}\, d\Omega$$

♡

Using the divergence theorem for the term ♡ we have:

$$\nu\int_{\partial\Omega}\underline{n}\cdot\left[(\underline{\nabla}\times\underline{u})\times\delta\underline{u}\right] dS + \nu\int_{\Omega}(\underline{\nabla}\times\delta\underline{u})\cdot(\underline{\nabla}\times\underline{u})\, d\Omega = \int_{\Omega}\underline{f}\cdot\delta\underline{u}\, d\Omega$$

$\forall\ \delta\underline{u}:\ \underline{n}\times\delta\underline{u} = \underline{0}$

knowing that: $\underline{a}\cdot(\underline{b}\times\underline{c}) = (\underline{a}\times\underline{b})\cdot\underline{c}$

we obtain:

$$\nu\int_{\partial\Omega}\left[\underline{n}\times(\underline{\nabla}\times\underline{u})\right]\cdot\delta\underline{u}\, dS + \nu\int_{\Omega}(\underline{\nabla}\times\delta\underline{u})\cdot(\underline{\nabla}\times\underline{u})\, d\Omega = \int_{\Omega}\delta\underline{u}\cdot\underline{f}\, d\Omega$$

the solution $\underline{u}$ must satisfy the following bounded conditions:

1) $\int_{\Omega}\|\underline{u}\|^2 d\Omega < \infty \quad \underline{u}\in L_2(\Omega)$

2) $\int_{\Omega}\|\underline{\nabla}\times\underline{u}\|^2 d\Omega < \infty\ ,\ \underline{\nabla}\times\underline{u}\in L_2(\Omega)$

We arrive to obtain the following conclusion:

$\Rightarrow$ $\underline{u} \in H(curl, \Omega)$ where

$$H(curl, \Omega) = \left\{ \underline{u} \in \Omega \rightarrow \mathbb{R}^3 : \int_\Omega |\underline{u}|^2 d\Omega < \infty, \int_\Omega |\underline{\nabla} \times \underline{u}|^2 d\Omega < \infty \right\}$$

but given the boundary condition ($\underline{n} \times \underline{u} = \underline{0}$ on $\partial\Omega$)

then $\underline{u} \in H_0(curl, \Omega) = \left\{ \underline{u} \in H(curl, \Omega) : \underline{n} \times \underline{u} = \underline{0} \text{ on } \partial\Omega \right\}$

Question (b)

The regularity requirement states that $\int_\Omega |\underline{\nabla} \times \underline{u}|^2 d\Omega < \infty$

the following :

Considering

$$\int_\Gamma \underline{n}_s \cdot (\underline{\nabla} \times \underline{u}) \, dS$$

Using the Stokes equation:

$$\int_\Gamma \underline{n} \cdot (\underline{\nabla} \times \underline{u}) \, dS = \int_{\partial\Gamma} \underline{u} \cdot \underline{t} \, dl = \int_{\partial\Gamma} \underline{n} \times \underline{u} \, dl$$

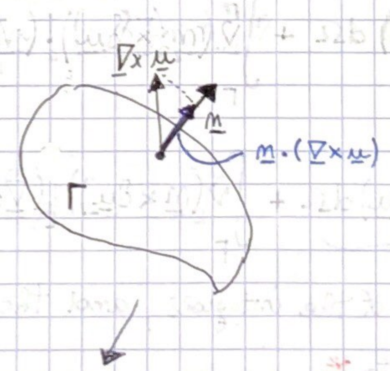
Since the problem is 3D, the interface is represented by a surface.

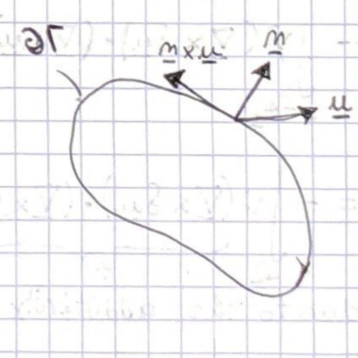
$\downarrow$ Since $\underline{n} \cdot (\underline{\nabla} \times \underline{u})$ is tangential as seen,

The jump for the tangential component of the unknown across the interface must be equal to zero

$\downarrow$

$[\![\underline{n} \times \underline{u}]\!] = \underline{0}$





The whole problem in our case considers the interface as a surface

$$\left(\int_\Gamma (\;) \, dS \right)$$

Question (C)

$\forall \delta \underline{u}$ such that :

$$\int_{\partial\Omega} \nu[\underline{n} \times (\underline{\nabla} \times \underline{u})] \cdot \delta\underline{u} \, dS + \nu \int_{\Omega} (\underline{\nabla} \times \delta\underline{u}) \cdot (\underline{\nabla} \times \underline{u}) \, d\Omega = \int_{\Omega} \delta\underline{u} \cdot \underline{f} \, d\Omega$$

knowing that $\underline{n} \times \delta\underline{u} = \underline{0}$ on $\partial\Omega$

focusing on one subdomain :

this is equal to : $\int_{\Omega_1} \delta\underline{u} \cdot (\nu \underline{\nabla} \times \underline{\nabla} \times \underline{u}) \, d\Omega = \nu \int_{\Omega_1} \underline{\nabla} \cdot ((\underline{\nabla} \times \underline{u}) \times \delta\underline{u}) \, d\Omega$

$$+ \nu \int_{\Omega_1} (\underline{\nabla} \times \delta\underline{u}) \cdot (\underline{\nabla} \times \underline{u}) \, d\Omega =$$

using divergence theorem (and the previous property for mixed product) : $\nu \int_{\partial\Omega_1} [\underline{n}_1 \times (\underline{\nabla} \times \underline{u})] \cdot \delta\underline{u} \, dS + \nu \int_{\Omega} (\underline{\nabla} \times \delta\underline{u}) \cdot (\underline{\nabla} \times \underline{u}) \, d\Omega$

the same thing can be written for Ω_2 : using additivity :

$$\int_{\partial\Omega} \nu [\underline{n} \times (\underline{\nabla} \times \underline{u})] \cdot \delta\underline{u} + \nu \underbrace{\int_{\Omega} (\underline{\nabla} \times \delta\underline{u}) \cdot (\underline{\nabla} \times \underline{u}) \, d\Omega}_{*} = \int_{\partial\Omega_1} \nu \, [\underline{n}_1 \times (\underline{\nabla} \times \underline{u})] \cdot \delta\underline{u} \, dS$$

$$+ \underbrace{\int_{\Omega_1} \nu \, (\underline{\nabla} \times \delta\underline{u}) \cdot (\underline{\nabla} \times \underline{u}) \, d\Omega}_{*} + \nu \int_{\partial\Omega_2} [\underline{n}_2 \times (\underline{\nabla} \times \underline{u})] \cdot \delta\underline{u} \, dS + \nu \underbrace{\int_{\Omega_2} (\underline{\nabla} \times \delta\underline{u}) \cdot (\underline{\nabla} \times \underline{u}) \, d\Omega}_{**}$$

We have the equality between $*$ and $(* + **)$. Only the boundary terms remain :

$$\int_{\partial\Omega} \nu [\underline{n} \times (\underline{\nabla} \times \underline{u})] \cdot \delta\underline{u} \, dS = \int_{\partial\Omega_1} \nu [\underline{n}_1 \times (\underline{\nabla} \times \underline{u})] \cdot \delta\underline{u} + \int_{\partial\Omega_2} \nu [\underline{n}_2 \times (\underline{\nabla} \times \underline{u})] \cdot \delta\underline{u}$$

only interface $\Rightarrow$ $(\underline{n} = \underline{n}_1 = -\underline{n}_2)$, $\nu = \text{cst}$

$$\int_{\Gamma} [\underline{n} \times (\underline{\nabla} \times \underline{u})|_1] \cdot \delta\underline{u} \, dS = \int_{\Gamma} [\underline{n} \times (\underline{\nabla} \times \underline{u})|_2] \cdot \delta\underline{u} \, dS \quad , \forall \delta\underline{u} \in H_0(\text{curl}, \Omega)$$

Second transmission condition

the same weak trasmission condition can be written as

$$[\![\underline{n} \times (\underline{\nabla} \times \underline{u})]\!]_\Gamma = 0 \quad \text{iff } \underline{u} \text{ is regular enough}$$

Case 3 : ELASTICITY NAVIER EQUATIONS:

[1] $-2\mu \, \underline{\nabla} \cdot (\underline{\nabla}^s \underline{u}) - \lambda \underline{\nabla}(\underline{\nabla} \cdot \underline{u}) = \rho \underline{b}$

[2] $-\mu \Delta \underline{u} - (\lambda + \mu) \underline{\nabla}(\underline{\nabla} \cdot \underline{u}) = \rho \underline{b}$

[3] $\mu \underline{\nabla} \times \underline{\nabla} \times \underline{u} - (\lambda + 2\mu) \underline{\nabla}(\underline{\nabla} \cdot \underline{u}) = \rho \underline{b}$ $\qquad \underline{\nabla}^s \underline{u} = \frac{1}{2}(\underline{\nabla} \underline{u} + \underline{\nabla} \underline{u}^T)$

where $\underline{u}$ is the dispeacement field, $\underline{\nabla}^s \underline{u}$ the symmetric part of $\underline{\nabla} \underline{u}$, λ and μ the Lamé coeff.s, ρ the density of the material and $\underline{b}$ the body forces

B.C. : $\underline{u} = \underline{0}$ on $\partial\Omega$

Question (a) - [1]

let's use $\delta\underline{u}$ such that $\delta\underline{u} = \underline{0}$ on $\partial\Omega$

$$\int_\Omega \delta\underline{u} \cdot (-2\mu \underline{\nabla} \cdot (\underline{\nabla}^s \underline{u}) - \lambda \underline{\nabla}(\underline{\nabla} \cdot \underline{u})) \, d\Omega = \int_\Omega \delta\underline{u} \cdot \rho \underline{b} \, d\Omega$$

$$\int_\Omega \delta\underline{u} \cdot (-2\mu \underline{\nabla} \cdot (\underline{\nabla}^s \underline{u}) - \lambda \underline{\nabla}(\underline{\nabla} \cdot \underline{u})) \, d\Omega = -\int_\Omega \underline{\nabla} \cdot (2\mu \, \delta\underline{u} [\underline{\nabla}^s \underline{u}]) \, d\Omega \, +$$

$$+ \, 2\mu \int_\Omega (\underline{\nabla}^s \delta\underline{u}) : (\underline{\nabla}^s \underline{u}) \, d\Omega - \int_\Omega \underline{\nabla} \cdot (\lambda (\underline{\nabla} \cdot \underline{u}) \delta\underline{u}) \, d\Omega + \int_\Omega \lambda (\underline{\nabla} \cdot \delta\underline{u})(\underline{\nabla} \cdot \underline{u}) \, d\Omega =$$

$$= 2\mu \int_\Omega (\underline{\nabla}^s \delta\underline{u}) : (\underline{\nabla}^s \underline{u}) \, d\Omega + \lambda \int_\Omega (\underline{\nabla} \cdot \delta\underline{u})(\underline{\nabla} \cdot \underline{u}) \, d\Omega - \int_{\partial\Omega} 2\mu ([\underline{\nabla}^s \underline{u}] \underbrace{\cancel{\delta\underline{u}}}_{0}) \cdot \underline{n} \, ds$$

(boundary conditions)

$$- \int_{\partial\Omega} \lambda ([\underline{\nabla} \cdot \underline{u}] \underbrace{\cancel{\delta\underline{u}}}_{0}) \cdot \underline{n} \, ds$$

(boundary conditions)

this gives us the final variational formulation:

$$\int 2\mu (\underline{\nabla}^s \delta\underline{u}) : (\underline{\nabla}^s \underline{u}) \, d\Omega + \lambda \int_\Omega (\underline{\nabla} \cdot \delta\underline{u})(\underline{\nabla} \cdot \underline{u}) \, d\Omega = \int_\Omega \delta\underline{u} \cdot \rho \underline{b} \, d\Omega$$

In this case the $\underline{u}$, $\underline{\nabla} \underline{u}$ and $\underline{\nabla} \cdot \underline{u}$ belongs to $L^2(\Omega)$,

$$L^2(\Omega) = \left\{ \underline{u} \in \Omega \rightarrow \mathbb{R}^3 : \int_\Omega |\underline{u}|^2 d\Omega < \infty \right\}$$

the space of function

$$H^1(\Omega) = \left\{ \underline{u} \in \Omega \to \mathbb{R}^3 : \int_\Omega |\underline{u}|^2 d\Omega < \infty, \int_\Omega |\underline{\nabla}\,\underline{u}|^2 d\Omega < \infty \right\}$$

and $H(div, \Omega) = \left\{ \underline{u} \in \Omega : \int_\Omega |\underline{u}|^2 d\Omega < \infty, \int_\Omega (\underline{\nabla} \cdot \underline{u})^2 d\Omega < \infty \right\}$

are the involved ones.

Since the divergence bounded is a ~~stronger~~ more particular case than the gradient bounded, the space $H(div, \Omega) \subset H^1(\Omega)$

then the proper space is $H^1(\Omega)$, with $\underline{u} = 0$ on $\partial\Omega \to H^1_0(\Omega)$

Question (b) - [1]

$\forall\, \delta\underline{u}$ such that

$$\int_\Omega 2\mu (\underline{\nabla}^s \delta\underline{u}) : (\underline{\nabla}^s \underline{u})\, d\Omega + \int_\Omega \lambda (\underline{\nabla} \cdot \delta\underline{u})(\underline{\nabla} \cdot \underline{u})\, d\Omega = \int_\Omega \delta\underline{u} \cdot \underline{f}^b d\Omega$$

knowing that $\delta\underline{u} = \underline{0}$ on $\partial\Omega$

Let us focus on one subdomain Ω_1: $\Omega = \Omega_1 \cup \Omega_2$

$$\int_{\Omega_1} \delta\underline{u} \cdot (-2\mu \underline{\nabla} \cdot (\underline{\nabla}^s \underline{u}))\, d\Omega + \int_{\Omega_1} \lambda\, \delta\underline{u} \cdot [\underline{\nabla}(\underline{\nabla} \cdot \underline{u})]\, d\Omega = \int_{\Omega_1} \underline{\nabla} \cdot (-2\mu\, \delta\underline{u} [\underline{\nabla}^s \underline{u}])\, d\Omega$$

$$+ \int_{\Omega_1} 2\mu (\underline{\nabla}^s \delta\underline{u}) : (\underline{\nabla}^s \underline{u})\, d\Omega - \int_{\Omega_1} \underline{\nabla} \cdot (\lambda [\underline{\nabla} \cdot \underline{u}]\, \delta\underline{u})\, d\Omega + \int_{\Omega_1} \lambda (\underline{\nabla} \cdot \delta\underline{u})(\underline{\nabla} \cdot \underline{u})\, d\Omega =$$

$$= -\int_{\cancel{\partial\Omega_1}\, \Gamma} \underline{n}_1 \cdot (2\mu\, \delta\underline{u} [\underline{\nabla}^s \underline{u}])\, dS - \int_{\cancel{\partial\Omega_1}\, \Gamma} \underline{n}_1 \cdot (\lambda [\underline{\nabla} \cdot \underline{u}]\, \delta\underline{u})\, dS + \int_{\Omega_1} 2\mu (\underline{\nabla}^s \delta\underline{u}) : (\underline{\nabla}^s \underline{u})\, d\Omega$$

(the rest is $= 0$) (the rest is $= 0$)

$$+ \int_{\Omega_1} \lambda (\underline{\nabla} \cdot \delta\underline{u})(\underline{\nabla} \cdot \underline{u})\, d\Omega$$

The same thing can be written for Ω_2

Let's use the additivity:

$$\left.\begin{aligned} &\int_{\Omega_1} \delta\underline{u} \cdot (-2\mu \underline{\nabla} \cdot (\underline{\nabla}^s \underline{u}))\, d\Omega + \int_{\Omega_1} \delta\underline{u} \cdot (\lambda \underline{\nabla} (\underline{\nabla} \cdot \underline{u}))\, d\Omega + \int_{\Omega_2} \delta\underline{u} \cdot (-2\mu \underline{\nabla} \cdot (\underline{\nabla}^s \underline{u}))\, d\Omega \\ &+ \int_{\Omega_2} \delta\underline{u} \cdot (-\lambda \underline{\nabla} (\underline{\nabla} \cdot \underline{u}))\, d\Omega = \end{aligned}\right\} *$$

$$= \int_\Gamma -\underline{n}_1 \cdot \left[(2\mu \delta\underline{u} [\underline{\nabla}^s \underline{u}] \right] \Big|_1 dS + \int_\Gamma -\underline{n}_1 \cdot \left[\lambda (\underline{\nabla} \cdot \underline{u}) \delta u \right] \Big|_1 dS + 2\mu \int_{\Omega_1} (\underline{\nabla}^s \delta\underline{u}) : (\underline{\nabla}^s \underline{u}) \, d\Omega$$

$$+ \lambda \int_{\Omega_1} (\underline{\nabla} \cdot \delta\underline{u})(\underline{\nabla} \cdot \underline{u}) \, d\Omega - \int_\Gamma \underline{n}_2 \cdot \left[2\mu \, \delta\underline{u} [\underline{\nabla}^s \underline{u}] \right] \Big|_2 dS + \int_\Gamma -\underline{n}_2 \cdot \left[\lambda (\underline{\nabla} \cdot \underline{u}) \, \delta\underline{u} \right] \Big|_2 dS$$

$$+ 2\mu \int_{\Omega_2} (\underline{\nabla}^s \delta\underline{u}) : (\underline{\nabla}^s \underline{u}) \, d\Omega + \lambda \int_{\Omega_2} (\underline{\nabla} \cdot \delta\underline{u})(\underline{\nabla} \cdot \underline{u}) \, d\Omega =$$

Using additivity, and using the fact that the sum is equal to the integrals *, we arrive to the following relation:

$$- \int_\Gamma \underline{n} \cdot \left[(2\mu (\underline{\nabla}^s \underline{u})) \Big|_1 - (2\mu (\underline{\nabla}^s \underline{u})) \Big|_2 \right] dS - \int_\Gamma \underline{n} \cdot \left[\lambda (\underline{\nabla} \cdot \underline{u}) \Big|_1 - \lambda (\underline{\nabla} \cdot \underline{u}) \Big|_2 \right] dS = \emptyset$$

$(\underline{n}_1 = \underline{n} = -\underline{n}_2)$

this brings to two transmission conditions:

$$* \quad \bullet \; 2\mu \int_\Gamma \underline{n} \cdot \left[(\underline{\nabla}^s \underline{u}) \Big|_1 - (\underline{\nabla}^s \underline{u}) \Big|_2 \right] \delta\underline{u} \, dS = 0 \quad \forall \delta\underline{u} \in H^1_0(\Omega)$$

$$** \quad \bullet \; \lambda \int_\Gamma \underline{n} \cdot \left[(\underline{\nabla} \cdot \underline{u}) \Big|_1 - (\underline{\nabla} \cdot \underline{u}) \Big|_2 \right] \delta\underline{u} \, dS = 0 \quad \forall \delta\underline{u} \in H^1_0(\Omega)$$

Question (a) - [2]

let us use $\delta\underline{u}$ such that:

$$\int_\Omega \delta\underline{u} \cdot (-\mu \Delta \underline{u} - (\mu + \lambda) \underline{\nabla} (\underline{\nabla} \cdot \underline{u})) \, d\Omega = \int_\Omega \delta\underline{u} \cdot \rho \underline{b}$$

and $\delta\underline{u} = \underline{0}$ on $\partial\Omega$

$$\int_\Omega \delta\underline{u} \cdot (-\mu \Delta \underline{u} - (\mu + \lambda) \underline{\nabla} (\underline{\nabla} \cdot \underline{u})) \, d\Omega = -\mu \int_\Omega \underline{\nabla} \cdot (\delta\underline{u} [\underline{\nabla} \underline{u}]) \, d\Omega +$$

$$+ \int_\Omega (\mu + \lambda) (\underline{\nabla} \cdot \delta\underline{u})(\underline{\nabla} \cdot \underline{u}) \, d\Omega + \int_\Omega \mu \, (\underline{\nabla} \delta\underline{u}) : (\underline{\nabla} \underline{u}) \, d\Omega - \int_\Omega \underline{\nabla} \cdot ((\lambda + \mu) [\underline{\nabla} \cdot \underline{u}] \, \delta\underline{u}) \, d\Omega$$

$$= \mu \int_{\Omega} (\underline{\nabla} \delta \underline{u}) : (\underline{\nabla} \underline{u}) \, d\Omega + (\mu+\lambda) \int_{\Omega} (\underline{\nabla} \cdot \delta \underline{u})(\underline{\nabla} \cdot \underline{u}) \, d\Omega -$$

$$- \int_{\partial\Omega} \underline{n} \cdot (\mu \, \cancel{\delta \underline{u}} [\underline{\nabla} \underline{u}]) \, dS - \int_{\Omega} \underline{n} \cdot \left[(\mu+\lambda) \cancel{\delta \underline{u}} [\underline{\nabla} \cdot \underline{u}] \right] dS$$

$\to 0$ $\to 0$

(due to the boundary conditions)

$$\Downarrow$$

$$\boxed{\mu \int_{\Omega} (\underline{\nabla} \delta \underline{u}) : (\underline{\nabla} \underline{u}) \, d\Omega + (\mu+\lambda) \int_{\Omega} (\underline{\nabla} \cdot \delta \underline{u})(\underline{\nabla} \cdot \underline{u}) \, d\Omega = \int_{\Omega} \delta \underline{u} \cdot \rho \underline{b} \, d\Omega}$$

In this case too, $\underline{u}$, $\underline{\nabla} \underline{u}$, $\underline{\nabla} \cdot \underline{u}$ must be $\in L^2(\Omega)$, with $\underline{u} = \underline{0}$ on $\partial\Omega$

So $\underline{u} \in H_1^{\circ}(\Omega)$ $\left[H_1(\Omega) \supset H(\text{div}, \Omega) \text{, see above} \right]$

Question (b) - [2]

$\forall \, \delta \underline{u}$ such that

$$\int_{\Omega} \mu (\underline{\nabla} \delta \underline{u}) : (\underline{\nabla} \underline{u}) \, d\Omega + (\mu+\lambda) \int_{\Omega} (\underline{\nabla} \cdot \delta \underline{u})(\underline{\nabla} \cdot \underline{u}) \, d\Omega = \int_{\Omega} \delta \underline{u} \cdot \rho \underline{b} \, d\Omega$$

Knowing that $\delta \underline{u} = \underline{0}$ on $\partial\Omega$

Let us focus on one sub domain Ω_1: $\Omega = \Omega_1 \cup \Omega_2$

$$\int_{\Omega_1} \delta \underline{u} \cdot (-\mu \Delta \underline{u}) \, d\Omega + \int_{\Omega_1} \delta \underline{u} \cdot (-(\mu+\lambda) \underline{\nabla}(\underline{\nabla} \cdot \underline{u})) \, d\Omega = -\mu \int_{\Omega_1} \underline{\nabla} \cdot (\delta \underline{u} [\underline{\nabla} \underline{u}]) \, d\Omega$$

$$+ \int_{\Omega_1} \mu (\underline{\nabla} \delta \underline{u}) : (\underline{\nabla} \underline{u}) \, d\Omega - \int_{\Omega_1} \underline{\nabla} \cdot ((\lambda+\mu) [\underline{\nabla} \cdot \underline{u}] \delta \underline{u}) \, d\Omega + \int_{\Omega_1} (\lambda+\mu)(\underline{\nabla} \cdot \delta \underline{u})(\underline{\nabla} \cdot \underline{u}) \, d\Omega =$$

$$= - \int_{\cancel{\partial\Omega_1} \Gamma} \underline{n}_1 \cdot (\mu \, \delta \underline{u} [\underline{\nabla} \underline{u}]) \, dS - \int_{\cancel{\partial\Omega_1} \Gamma} \underline{n}_1 \cdot ((\lambda+\mu) [\underline{\nabla} \cdot \underline{u}] \delta \underline{u}) \, dS + \int_{\Omega_1} \mu (\underline{\nabla} \delta \underline{u}) : (\underline{\nabla} \underline{u}) \, d\Omega$$

Γ (the rest is $= \emptyset$) Γ (the rest is $= \emptyset$)

$$+ \int_{\Omega_1} (\lambda+\mu)(\underline{\nabla} \cdot \delta \underline{u})(\underline{\nabla} \cdot \underline{u}) \, d\Omega$$

The same thing can be written for Ω_2

Let's use the additivity :

$$\underbrace{\int_{\Omega_1} \delta\underline{u} \cdot (-\mu \Delta \underline{u})\, d\Omega + \int_{\Omega_1} \delta\underline{u} \cdot \left(-(\lambda+\mu)\, \underline{\nabla}(\underline{\nabla}\cdot \underline{u})\right) d\Omega + \int_{\Omega_2} \delta\underline{u} \cdot (-\mu \Delta \underline{u})\, d\Omega + \int_{\Omega_2} \delta\underline{u} \cdot \left(-(\lambda+\mu)\, \underline{\nabla}(\underline{\nabla}\cdot \underline{u})\right) d\Omega}_{*} =$$

$$= - \int_{\Gamma} \underline{n}_1 \cdot (\mu\, \delta\underline{u}\, [\underline{\nabla}\, \underline{u}])\, dS - \int_{\Gamma} \underline{n}_1 \cdot ((\lambda+\mu)[\underline{\nabla}\cdot \underline{u}]\, \delta\underline{u})\, dS +$$

$$+ \int_{\Omega_1} \mu\, (\underline{\nabla}\, \delta\underline{u}) : (\underline{\nabla}\, \underline{u})\, d\Omega + \int_{\Omega_1} (\lambda+\mu)(\underline{\nabla}\cdot \delta\underline{u})(\underline{\nabla}\cdot \underline{u})\, d\Omega -$$

$$- \int_{\Gamma} \underline{n}_2 \cdot (\mu\, \delta\underline{u}\, [\underline{\nabla}\, \underline{u}])\, dS - \int_{\Gamma} \underline{n}_2 \cdot ((\lambda+\mu)[\underline{\nabla}\cdot \underline{u}]\, \delta\underline{u})\, dS +$$

$$+ \int_{\Omega_2} \mu\, (\underline{\nabla}\, \delta\underline{u}) : (\underline{\nabla}\, \underline{u})\, d\Omega + \int_{\Omega_2} (\lambda+\mu)(\underline{\nabla}\cdot \delta\underline{u})(\underline{\nabla}\cdot \underline{u})\, d\Omega \,.$$

Using additivity, and the following relation: $\underline{n} = \underline{n}_1 = -\underline{n}_2$

$$= \underbrace{\int_{\Omega} \mu\, (\underline{\nabla}\delta\underline{u}) : (\underline{\nabla}\, \underline{u})\, d\Omega + \int_{\Omega} (\lambda+\mu)(\underline{\nabla}\cdot \delta\underline{u})(\underline{\nabla}\cdot \underline{u})\, d\Omega}_{*} -$$

$$- \int_{\Gamma} \underline{n} \cdot \left[(\mu\, \delta\underline{u}\, [\underline{\nabla}\underline{u}])\Big|_1 - (\mu\, \delta\underline{u}\, [\underline{\nabla}\, \underline{u}])\Big|_2 \right] dS - \int_{\Gamma} \underline{n} \cdot \left[((\lambda+\mu)[\underline{\nabla}\cdot\underline{u}]\, \delta\underline{u}\Big|_1 - (\lambda+\mu)[\underline{\nabla}\cdot\underline{u}]\, \delta\underline{u}\Big|_2 \right] dS$$

Imposing that * must be equal to *, we obtain

$$\int_{\Gamma} \underline{n} \cdot \left(\left[\mu [\underline{\nabla}\, \underline{u}]\Big|_1 - \mu [\underline{\nabla}\, \underline{u}]\Big|_2 \right] \delta\underline{u} \right) dS = 0 \qquad \forall\, \delta\underline{u} \in H^1_0(\Omega)$$

$$\int_{\Gamma} \underline{n} \cdot \left[\left((\lambda+\mu)[\underline{\nabla}\cdot \underline{u}]\Big|_1 - (\lambda+\mu)[\underline{\nabla}\cdot \underline{u}]\Big|_2 \right) \delta\underline{u} \right] dS = 0 \quad \forall\, \delta\underline{u} \in H^1_0(\Omega)$$

which represent the two transmission conditions.

Question (a) - [3]

let us use $\delta \underline{u}$ such that:

$$\int_\Omega \delta\underline{u}\cdot(\mu\underline{\nabla}\times\underline{\nabla}\times\underline{u})\,d\Omega - (\lambda+2\mu)\int_\Omega \delta\underline{u}\,[\underline{\nabla}(\underline{\nabla}\cdot\underline{u})]\,d\Omega = \int_\Omega \delta\underline{u}\cdot\rho\underline{b}\,d\Omega$$

$$\int_\Omega \delta\underline{u}\cdot(\mu\underline{\nabla}\times\underline{\nabla}\times\underline{u})\,d\Omega - \int_\Omega(\lambda+2\mu)\,\delta\underline{u}\,\underline{\nabla}(\underline{\nabla}\cdot\underline{u})\,d\Omega = \int_\Omega \underline{\nabla}\cdot(\mu\,\delta\underline{u}\times(\underline{\nabla}\times\underline{u}))\,d\Omega$$

$$+\int_\Omega \mu(\underline{\nabla}\times\delta\underline{u})\cdot(\underline{\nabla}\times\underline{u})\,d\Omega - \int_\Omega \underline{\nabla}\cdot((\lambda+\mu)[\underline{\nabla}\cdot\underline{u}]\,\delta\underline{u})\,d\Omega +$$

$$+\int_\Omega(\lambda+2\mu)(\underline{\nabla}\cdot\delta\underline{u})(\underline{\nabla}\cdot\underline{u})\,d\Omega =$$

$$= \int_{\partial\Omega}\mu\,[\underbrace{\underline{n}\times(\underline{\nabla}\times\underline{u})}_{\to 0 \text{ (given the B.C)}}]\cdot\delta\underline{u}\,dS - \int_{\partial\Omega}(\lambda+2\mu)\,\underline{n}\cdot(\underbrace{[\underline{\nabla}\cdot\underline{u}]}_{\to 0 \text{ (given the Boundary Conditions)}}\,\delta\underline{u})\,dS \; -$$

$$+\mu\int_\Omega(\underline{\nabla}\times\delta\underline{u})\cdot(\underline{\nabla}\times\underline{u})\,d\Omega + \int_\Omega(\lambda+2\mu)(\underline{\nabla}\cdot\delta\underline{u})(\underline{\nabla}\cdot\underline{u})\,d\Omega$$

We obtain the variational formulation:

$$+\mu\int_\Omega(\underline{\nabla}\times\delta\underline{u})\cdot(\underline{\nabla}\times\underline{u})\,d\Omega + \int_\Omega(\lambda+2\mu)(\underline{\nabla}\cdot\delta\underline{u})(\underline{\nabla}\cdot\underline{u})\,d\Omega = \int_\Omega \delta\underline{u}\cdot\rho\underline{b}\,d\Omega$$

In this problem $\underline{u} \in H(div,\Omega) \cup H(curl,\Omega) \equiv H^1_0(\Omega)$

where $H(div,\Omega) = \left\{ \underline{u} \in \Omega : \int_\Omega |\underline{u}|^2 d\Omega < \infty,\ \int_\Omega [\underline{\nabla}\cdot\underline{u}]^2 d\Omega < \infty \right\}$

Since the B.C. say that $\underline{u} = \underline{0}$ on $\partial\Omega$, $\underline{u} \in H^1_0(\Omega)$

Question (b) - [3]

$\forall\,\delta\underline{u}$ such that:

$$\int_\Omega +\mu(\underline{\nabla}\times\delta\underline{u})\cdot(\underline{\nabla}\times\underline{u})\,d\Omega + \int_\Omega(\lambda+2\mu)(\underline{\nabla}\cdot\delta\underline{u})(\underline{\nabla}\cdot\underline{u})\,d\Omega = \int_\Omega \delta\underline{u}\cdot\rho\underline{b}\,d\Omega$$

knowing that $\delta \underline{u} = \underline{0}$ on $\partial\Omega$

Let us focus on one subdomain Ω_1: $\Omega = \Omega_1 \cup \Omega_2$

$$\int_{\Omega_1} \delta\underline{u} \cdot (\mu \underline{\nabla} \times \underline{\nabla} \times \underline{u})\, d\Omega - (\lambda+\mu) \int_{\Omega_1} \delta\underline{u} \cdot [\underline{\nabla}(\underline{\nabla}\cdot\underline{u})]\, d\Omega = \int_{\Omega_1} \underline{\nabla} \cdot (\mu\, \delta\underline{u} \times (\underline{\nabla}\times\underline{u}))\, d\Omega$$

$$+ \int_{\Omega_1} \mu (\underline{\nabla}\times\delta\underline{u}) \cdot (\underline{\nabla}\times\underline{u})\, d\Omega - \int_{\Omega_1} \underline{\nabla}\cdot((\lambda+\mu)[\underline{\nabla}\cdot\underline{u}]\delta\underline{u})\, d\Omega +$$

$$+ \int_{\Omega_1} (\lambda+2\mu)(\underline{\nabla}\cdot\delta\underline{u})(\underline{\nabla}\cdot\underline{u})\, d\Omega =$$

$$= \int_{\cancel{\partial\Omega_1}\ \Gamma\ (\text{the rest is } = \emptyset)} \mu [\underline{n}_1 \times (\underline{\nabla}\times\underline{u})] \cdot \delta\underline{u}\, dS - \int_{\cancel{\partial\Omega_1}\ \Gamma\ (\text{the rest is } \neq \emptyset)} (\lambda+2\mu)\, \underline{n}_1 \cdot ([\underline{\nabla}\cdot\underline{u}]\delta\underline{u}) + \int_{\Omega_1} \mu (\underline{\nabla}\times\delta\underline{u})\cdot(\underline{\nabla}\times\underline{u})\, d\Omega$$

$$+ \int_{\Omega_1} (\lambda+2\mu)(\underline{\nabla}\cdot\delta\underline{u})(\underline{\nabla}\cdot\underline{u})\, d\Omega$$

The same can be written for Ω_2

Let us use the additivity:

$$\int_{\Omega_1} \delta\underline{u}\cdot(\mu\underline{\nabla}\times\underline{\nabla}\times\underline{u})\, d\Omega - (\lambda+\mu)\int_{\Omega_1} \delta\underline{u}\cdot[\underline{\nabla}(\underline{\nabla}\cdot\underline{u})]\, d\Omega + \int_{\Omega_2} \delta\underline{u}\cdot(\mu\underline{\nabla}\times\underline{\nabla}\times\underline{u})\, d\Omega -$$

$$- (\lambda+\mu)\int_{\Omega_2} \delta\underline{u}\cdot[\underline{\nabla}(\underline{\nabla}\cdot\underline{u})]\, d\Omega =$$

$$= \int_{\Gamma} \mu[\underline{n}_1\times(\underline{\nabla}\times\underline{u})]\cdot\delta\underline{u}\, dS - \int_{\Gamma} (\lambda+2\mu)\, \underline{n}_1\cdot([\underline{\nabla}\cdot\underline{u}]\delta\underline{u})\, dS + \int_{\Omega_1} \mu(\underline{\nabla}\times\delta\underline{u})\cdot(\underline{\nabla}\times\underline{u})\, d\Omega$$

$$+ \int_{\Omega_1} (\lambda+2\mu)(\underline{\nabla}\cdot\delta\underline{u})(\underline{\nabla}\cdot\underline{u})\, d\Omega + \int_{\Gamma} \mu[\underline{n}_2\times(\underline{\nabla}\times\underline{u})]\cdot\delta\underline{u}\, dS - \int_{\Gamma} (\lambda+2\mu)\, \underline{n}_2\cdot([\underline{\nabla}\cdot\underline{u}]\delta\underline{u})\, dS$$

$$+ \int_{\Omega_2} \mu(\underline{\nabla}\times\delta\underline{u})\cdot(\underline{\nabla}\times\underline{u})\, d\Omega + \int_{\Omega_2} (\lambda+2\mu)(\underline{\nabla}\cdot\delta\underline{u})(\underline{\nabla}\cdot\underline{u})\, d\Omega$$

Using additivity, and the following relation: $\underline{n} = \underline{n}_1 = -\underline{n}_2$

$$= \underbrace{\int_\Omega +\mu(\underline{\nabla}\times\delta\underline{u})\cdot(\underline{\nabla}\times\underline{u})\,d\Omega + \int_\Omega (\lambda+2\mu)(\underline{\nabla}\cdot\delta\underline{u})(\underline{\nabla}\cdot\underline{u})\,d\Omega}_{*} +$$

$$+\int_\Gamma \underline{n}\times\left([\mu\underline{\nabla}\times\underline{u}]\Big|_1 - [\mu\underline{\nabla}\times\underline{u}]\Big|_2\right)\cdot\delta\underline{u}\,dS -$$

$$-\int_\Gamma \underline{n}\cdot\left((\lambda+2\mu)[\underline{\nabla}\cdot\underline{u}]\Big|_1 - (\lambda+2\mu)[\underline{\nabla}\cdot\underline{u}]\Big|_2\right)\delta\underline{u}\,dS$$

~~K~~nowing that $*$ must be equal to $*$, we obtain

$$\int_\Gamma \underline{n}\times\left([\mu\underline{\nabla}\times\underline{u}]\Big|_1 - [\mu\underline{\nabla}\times\underline{u}]\Big|_2\right)\cdot\delta\underline{u}\,dS = \emptyset \qquad \forall\,\delta\underline{u}\in H_0^1(\Omega)$$

$$\int_\Gamma \underline{n}\cdot\left((\lambda+2\mu)[\underline{\nabla}\cdot\underline{u}]\Big|_1 - (\lambda+2\mu)[\underline{\nabla}\cdot\underline{u}]\Big|_2\right)\delta\underline{u}\,dS = \emptyset \qquad \forall\,\delta\underline{u}\in H_0^1(\Omega)$$

Which are the two transmission conditions

Exercise 2 : DOMAIN DECOMPOSITION METHODS

Problem 1 : EULER - BERNOULLI BEAM

Let $[0,L] = [0,L_1]\cup[L_2,L]$, with $L_2 < L_1$ $\quad v(0) = v(L) = \emptyset$, $v'(0) = v'(L) = \emptyset$

Question (a)

The iteration by subdomain scheme based on a Schwartz additive Domain Decomposition Method, is to solve two Dirichlet problems

Sbd 1]	Sbd 2]
$EI\,\dfrac{d^4v_1^{(k)}}{dx^4} = f$ in Ω_1	$EI\,\dfrac{d^4v_2^{(k)}}{dx^4} = f$ in Ω_2
$v_1^{(k)} = 0$ on $x=0$	$v_2^{(k)} = 0$ on $x=0$
$\dfrac{dv_1^{(k)}}{dx} = 0$ on $x=0$	$\dfrac{dv_2^{(k)}}{dx} = 0$ on $x=0$
$v_1^{(k)} = v_2^{(k-1)}$ on $x=L_1$	$v_2^{(k)} = v_1^{(k-1)}$ on $x=L_2$
$\dfrac{dv_1^{(k)}}{dx} = \dfrac{dv_2^{(k-1)}}{dx}$ on $x=L_1$	$\dfrac{dv_2^{(k)}}{dx} = \dfrac{dv_1^{(k-1)}}{dx}$ on $x=L_2$

The trasmission condition is given by the previous iteration (Additive)

Question (b)

Once discretized in the FE space, the structural matrix for the subdomains is:

I: internal

L: interface

$$\begin{bmatrix} \underline{\underline{A}}_{I_1I_1} & \underline{\underline{A}}_{I_1L_1} \\ \underline{\underline{A}}_{L_1I_1} & \underline{\underline{A}}_{L_1L_1} \end{bmatrix} \begin{bmatrix} \underline{v}_{I_1} \\ \underline{v}_{L_1} \end{bmatrix} = \begin{bmatrix} \underline{f}_{I_1} \\ \underline{f}_{L_1} \end{bmatrix}$$

We solve the first question because the only unknown is the $\underline{v}_{I_1}$ and we assumed that the external B.C. has been imposed

STEP 1a →

$$\underline{\underline{A}}_{I_1I_1} \underline{v}_{I_1}^{(k)} + \underline{\underline{A}}_{I_1L_1} \underline{v}_{L_1}^{(k)} = \underline{f}_{I_1}$$

$$\underline{v}_{I_1}^{(k)} = \underline{\underline{A}}_{I_1I_1}^{-1} \left(\underline{f}_{I_1} - \underline{\underline{A}}_{I_1L_1} \underline{v}_{L_1}^{(k)} \right)$$

where $\underline{v}_{L_1}^{(k)} = \underline{v}_{I_2}^{(k-1)} \Big|_{L_1}$

The same thing can be done for the first derivatives (rotations):

$$\begin{bmatrix} \tilde{\underline{\underline{A}}}_{I_1I_1} & \tilde{\underline{\underline{A}}}_{I_1L_1} \\ \tilde{\underline{\underline{A}}}_{L_1I_1} & \tilde{\underline{\underline{A}}}_{L_1L_1} \end{bmatrix} \begin{bmatrix} \underline{v}'_{I_1} \\ \underline{v}'_{L_1} \end{bmatrix} = \begin{bmatrix} \tilde{\underline{f}}_{I_1} \\ \tilde{\underline{f}}_{L_1} \end{bmatrix}$$

(with external B.C. already composed)

STEP 1b →

$$\tilde{\underline{\underline{A}}}_{I_1I_1} \underline{v}'^{(k)}_{I_1} + \tilde{\underline{\underline{A}}}_{I_1L_1} \underline{v}'^{(k)}_{L_1} = \tilde{\underline{f}}_{I_1}$$

$$\underline{v}'^{(k)}_{I_1} = \tilde{\underline{\underline{A}}}_{I_1I_1}^{-1} \left(\tilde{\underline{f}}_{I_1} - \tilde{\underline{\underline{A}}}_{I_1L_1} \underline{v}'^{(k)}_{L_1} \right)$$

where $\underline{v}'^{(k)}_{L_1} = \underline{v}'^{(k-1)}_{I_2} \Big|_{L_1}$

The step 1a and step 1b are inside the first subdomain.

in the other subdomain Ω_2 the same thing can be done, in in a parallel form : (JACOBI METHOD)

STEP 2a

$$\underline{v}_{I_2}^{(k)} = \underline{\underline{A}}_{I_2 I_2}^{-1}\left(\underline{f}_{I_2} - \underline{\underline{A}}_{I_2 L_2}\underline{v}_{L_2}^{(k)}\right)$$

$$\underline{v}_{L_2}^{(k)} = \underline{v}_{I_1}^{(k-1)}\Big|_{L_2}$$

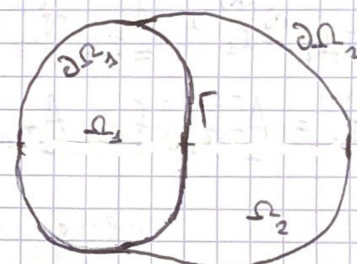
STEP 2b

$$\underline{v}_{I_2}^{\prime (k)} = \underline{\underline{\tilde{A}}}_{I_2 I_2}^{-1}\left(\underline{\tilde{f}}_{I_2} - \underline{\underline{\tilde{A}}}_{I_2 L_2}\underline{v}_{L_2}^{\prime (k)}\right)$$

$$\underline{v}_{L_2}^{\prime (k)} = \underline{v}_{I_1}^{(k-1)}\Big|_{L_2}$$

PROBLEM 2 : MAXWELL PROBLEM

$$\begin{cases} \nu\, \underline{\nabla}\times\underline{\nabla}\times\underline{u} = \underline{f} & \text{in } \Omega \\ \underline{\nabla}\cdot\underline{u} = 0 & \text{in } \Omega \\ \underline{n}\times\underline{u} = \underline{0} & \text{on } \partial\Omega \end{cases}$$



Question (a)

Dirichlet-Neumann coupling with IBS scheme

Sbd 1]

NEUMANN

$$\begin{cases} \nu\, \underline{\nabla}\times\underline{\nabla}\times\underline{u}_1^{(k)} = \underline{f}_1 & \text{in } \Omega_1 \\ \underline{\nabla}\cdot\underline{u}_1^{(k)} = 0 & \text{in } \Omega_1 \\ \underline{n}_1\times\underline{u}_1^{(k)} = \underline{0} & \text{on } \partial\Omega_1/\Gamma \\ \underline{n}_1\times(\underline{\nabla}\times\underline{u}_1^{(k)}) = \underline{n}_1\times(\underline{\nabla}\times\underline{u}_2^{k-1}) & \text{on } \Gamma \end{cases}$$

from Sbd 2

Sbd 2]

DIRICHLET

$$\begin{cases} \nu\, \underline{\nabla}\times\underline{\nabla}\times\underline{u}_2^{(k)} = \underline{f}_2 & \text{in } \Omega_2 \\ \underline{\nabla}\cdot\underline{u}_2^{(k)} = 0 & \text{in } \Omega_2 \\ \underline{n}_2\times\underline{u}_2^{(k)} = \underline{0} & \text{on } \partial\Omega_2/\Gamma \\ \underline{n}_2\times\underline{u}_2^{(k)} = \underline{n}_2\times\underline{u}_1^{(\ell)} & \text{on } \Gamma \end{cases}$$

if $\ell = k-1$ we have Jacobi scheme ; if $\ell = k$ we have Gauss-Seidel scheme

Question (b)

Given the following split: $\underline{u}_i = \underline{u}_i^0 + \underline{\tilde{u}}_i$, $i=1,2$

We have:
$$\begin{cases} \underline{\nabla} \times \underline{\nabla} \times \underline{u}_i^0 = \underline{f}_i & \text{in } \Omega_i \\ \underline{n}_i \times \underline{u}_i^0 = \underline{0} & \text{on } \partial\Omega_i / \Gamma \\ \underline{\nabla}_i \cdot \underline{u}_i^0 = 0 & \text{in } \Omega_i \\ \underline{n}_i \times \underline{u}_i^0 = \underline{0} & \text{on } \Gamma \end{cases}$$

and
$$\begin{cases} \nu \underline{\nabla} \times \underline{\nabla} \times \underline{\tilde{u}}_i = \underline{0} & \text{in } \Omega_i \\ \underline{\nabla}_i \cdot \underline{\tilde{u}}_i = 0 & \text{in } \Omega_i \\ \underline{n}_i \times \underline{\tilde{u}}_i = \underline{0} & \text{on } \partial\Omega_i / \Gamma \\ \underline{n}_i \times \underline{\tilde{u}}_i = \underline{\phi} & \text{on } \Gamma \end{cases}$$

Where the $\underline{u}_i^0$ depends only on the source $\underline{f}$, while $\underline{\tilde{u}}_i$ depends on the value at the interface $\underline{\phi}$. This problem, as already said, has the following transmission condition:

$$[\![\underline{n} \times (\underline{\nabla} \times \underline{u})]\!]_\Gamma = \underline{0}$$

So:
$$\underline{n} \times (\underline{\nabla} \times \underline{u}_1) = \underline{n} \times (\underline{\nabla} \times \underline{u}_2)$$

$$\underline{n} \times (\underline{\nabla} \times (\underline{u}_1^0 + \underline{\tilde{u}}_1)) = \underline{n} \times (\underline{\nabla} \times (\underline{u}_1^0 + \underline{\tilde{u}}_1))$$

$$\underline{n} \times (\underline{\nabla} \times \underline{\tilde{u}}_1) - \underline{n} \times (\underline{\nabla} \times \underline{\tilde{u}}_2) = \underline{n} \times (\underline{\nabla} \times \underline{u}_2^0) - \underline{n} \times (\underline{\nabla} \times \underline{u}_1^0)$$

So the first side depends on $\underline{\phi}$ while the RHS depends on the source $(\underline{G}) \rightarrow$ the whole transmission problem is shown as:

$$\boxed{\underline{\underline{S}}\,\underline{\phi} = \underline{G}}$$

where $\underline{\underline{S}}$ is the Steklov-Poincaré operator

Question (c) The matrix version of the whole domain discretized and partitioned

$$\begin{bmatrix} \underline{\underline{A}}_{II}^{(1)} & \underline{\underline{0}} & \underline{\underline{A}}_{I\Gamma}^{(1)} \\ \underline{\underline{0}} & \underline{\underline{A}}_{II}^{(2)} & \underline{\underline{A}}_{I\Gamma}^{(2)} \\ \underline{\underline{A}}_{\Gamma I}^{(1)} & \underline{\underline{A}}_{\Gamma I}^{(2)} & \underline{\underline{A}}_{\Gamma\Gamma}^{(1)} + \underline{\underline{A}}_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} \underline{u}_I^{(1)} \\ \underline{u}_I^{(2)} \\ \underline{u}_\Gamma \end{bmatrix} = \begin{bmatrix} \underline{f}_I^{(1)} \\ \underline{f}_I^{(2)} \\ \underline{f}_\Gamma^{(1)} + \underline{f}_\Gamma^{(2)} \end{bmatrix}$$

I: internal nodes

Γ: interface nodes

The Neumann problem in Sbd 1

$$\begin{bmatrix} \underline{\underline{A}}_{II}^{(1)} & \underline{\underline{A}}_{I\Gamma}^{(1)} \\ \underline{\underline{A}}_{\Gamma I}^{(1)} & \underline{\underline{A}}_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} \underline{u}_I^{(1)k} \\ \underline{u}_\Gamma^{k} \end{bmatrix} = \begin{bmatrix} \underline{f}_I \\ \underline{f}_\Gamma - \underline{\underline{A}}_{\Gamma I}^{(2)} \underline{u}_I^{(2)k-1} - \underline{\underline{A}}_{\Gamma\Gamma}^{(2)} \underline{u}_\Gamma^{k-1} \end{bmatrix}$$

NEUMANN BOUNDARY CONDITION

we get $\underline{u}_I^{(2)}$ from the previous scheme

The Dirichlet problem in Sbd 2

$$\underline{\underline{A}}_{II}^{(2)} \underline{u}_I^{(2)k} = \underline{f}_I^{(2)} - \underline{\underline{A}}_{I\Gamma}^{(2)} \underline{u}_\Gamma^{k-1}$$

DIRICHLET BOUNDARY CONDITION

PROBLEM 3 : POISSON PROBLEM

Finding $u: \Omega \to \mathbb{R}$ such that $\begin{cases} -k\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

where $k>0$. Let Γ be a surface crossing Ω

Question (a)

The Dirichlet problem in Ω_1:

$$\begin{cases} -k\Delta u_1^k = f_1 & \text{in } \Omega_1 \\ u_1 = 0 & \text{on } \partial\Omega_1/\Gamma \\ u_1^k = u_2^l & \text{on } \Gamma \end{cases}$$

The Robin problem in Ω_2:

$$\begin{cases} -k\Delta u_2^k = f_2 & \text{in } \Omega_2 \\ u_2 = 0 & \text{on } \partial\Omega_2/\Gamma \\ k\dfrac{\partial u_2^k}{\partial n} + \gamma_2 u_2^k = k\dfrac{\partial u_1^{k-1}}{\partial n} + \gamma_1 u_1^{k-1} & \text{on } \Gamma \end{cases}$$

Where $\ell = k-1$: Jacobi / Additive method

$\ell = k$: Gauss-Seidel / Multiplicative method

and $\gamma_1 + \gamma_2 > 0$

Question (b)

• Ω_1] Let us introduce δu_1 such that $\delta u_1 = 0$ on $\partial\Omega_1$
(test function)

$$\int_{\Omega_1} -\delta u_1 \cdot k \Delta u_1 \, d\Omega = \int_{\Omega_1} \delta u_1 f_1 \, d\Omega$$

$$= \int_{\Omega_1} -k\, \delta u_1 \cdot \Delta u_1 \, d\Omega = -k \int_{\Omega_1} \underline{\nabla} \cdot (\delta u_1 \underline{\nabla} u_1) \, d\Omega + \int_{\Omega_1} k \, \underline{\nabla} \delta u_1 \cdot \underline{\nabla} u_1 \, d\Omega$$

$$= -k \int_{\partial\Omega_1} \underline{n} \cdot (\underbrace{\delta u_1}_{0} \underline{\nabla} u_1) \, dS + \int_{\Omega_1} k (\underline{\nabla} \delta u_1 \cdot \underline{\nabla} u_1) \, d\Omega$$

(given by B.C.)

$$\int_{\Omega_1} k (\underline{\nabla} \delta u_1 \cdot \underline{\nabla} u_1) \, d\Omega = \int_{\Omega_1} \delta u_1 f_1 \, d\Omega$$

The same in matrix form says that:

$$\underline{\underline{A}}^{(1)}_{II} \underline{u}^{(1)k}_{I} = \underline{F}_1 - \underline{\underline{A}}^{(1)}_{I\Gamma} \underline{u}^{\ell}_{\Gamma}$$

• Ω_2] Introducing δu_2 such that $\delta u_2 = 0$ on $\partial\Omega_2$ (test function)

$$\int_{\Omega_2} -\delta u_2 \cdot k \Delta u_2 \, d\Omega = \int_{\Omega_2} \delta u_2 f_2 \, d\Omega$$

$$= \int_{\Omega_2} -k\, \delta u_2 \cdot \Delta u_2 \, d\Omega = -k \int_{\Omega_2} \underline{\nabla} \cdot (\delta u_2 \underline{\nabla} u_2) \, d\Omega + \int_{\Omega_2} k \, \underline{\nabla} \delta u_2 \cdot \underline{\nabla} u_2 \, d\Omega$$

$$= -k \int_{\partial\Omega_2} \underline{n} \cdot (\delta u_2 \underline{\nabla} u_2) \, dS + \int_{\Omega_2} k (\underline{\nabla} \delta u_2 \cdot \underline{\nabla} u_2) \, d\Omega$$

$$\int_{\Omega_2} k(\underline{\nabla}\delta u_2 \cdot \underline{\nabla} u_2)\, d\Omega - \int_{\Gamma} k\, \delta u_2 (\underline{n} \cdot \underline{\nabla} u_2)\, dS = \int_{\Omega_2} \delta u_2 f_2\, d\Omega$$

this is due to the fact that

$$\int_{\Gamma} k\, (\underline{n} \cdot \underline{\nabla} u_2)\, \delta u_2\, dS \neq 0$$, since it's the only

extra part that doesn't disappear.

Since $k(\underline{n} \cdot \underline{\nabla} u_2) = k \dfrac{\partial u_1}{\partial n} + \gamma_1 u_1 - \gamma_2 u_2 = k \dfrac{\partial u_2}{\partial n}$

$$\rightarrow \int_{\Omega_2} k(\underline{\nabla}\delta u_2 \cdot \underline{\nabla} u_2)\, d\Omega - \int_{\Gamma} k \frac{\partial u_2}{\partial n} \delta u_2\, dS = \int_{\Omega_2} \delta u_2 f_2\, d\Omega =$$

$$= \int_{\Omega_2} k(\underline{\nabla}\delta u_2 \cdot \underline{\nabla} u_2)\, d\Omega + \int_{\Gamma} k \gamma_2 \delta u_2 u_2\, dS = \int_{\Omega_2} \delta u_2 f_2\, d\Omega_2 +$$

$$+ \int_{\Gamma} k \gamma_1 \delta u_2 \frac{\partial u_1}{\partial n}\, dS + \int_{\Gamma} k \gamma_1 \delta u_2 u_1\, dS$$

$\Downarrow$

In the matrix form:

$$\begin{bmatrix} \underline{\underline{A}}^{(2)}_{II} & \underline{\underline{A}}^{(2)}_{I\Gamma} \\ \underline{\underline{A}}^{(2)}_{\Gamma I} + \gamma_2 \underline{\underline{M}}^{(2)}_{\Gamma I} & \underline{\underline{A}}^{(2)}_{\Gamma\Gamma} + \gamma_2 \underline{\underline{M}}^{(2)}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \underline{u}^{(2)^k}_I \\ \underline{u}^k_\Gamma \end{bmatrix} =$$

$$= \begin{bmatrix} \underline{f}^{(2)}_I \\ \underline{f}^{(1)}_I + \underline{f}^{(2)}_\Gamma - (\underline{\underline{A}}^{(1)}_{\Gamma I} - \gamma_1 \underline{\underline{M}}^{(1)}_{\Gamma I}) \underline{u}^{(1)^{k-1}}_I - (\underline{\underline{A}}^{(1)}_{\Gamma\Gamma} - \gamma_1 \underline{\underline{M}}^{(1)}_{\Gamma\Gamma}) \underline{u}^{k-1}_\Gamma \end{bmatrix}$$

Where I: Interior nodes , $\underline{\underline{A}}$ = stiffness matrix (FEM)

Γ: Interface nodes , $\underline{\underline{M}}$ = mass matrix (FEM)

Question (c)

- The first sbd problem: $\underline{u}_I^{(1)^k} = \underline{\underline{A}}_{II}^{(1)^{-1}} \left(\underline{f}_I^{(1)} - \underline{\underline{A}}_{I\Gamma}^{(1)} \underline{u}_\Gamma^{\ell} \right)$ [A]

- " second sbd problem: 1) $\underline{u}_I^{(2)^k} = \underline{\underline{A}}_{II}^{(2)^{-1}} \left(\underline{f}_I^{(2)} - \underline{\underline{A}}_{I\Gamma}^{(2)} \underline{u}_\Gamma^{k} \right)$ [B]

2) $\left(\hat{\underline{\underline{A}}}_{\Gamma\Gamma}^{(2)} - \hat{\underline{\underline{A}}}_{\Gamma I}^{(2)} \underline{\underline{A}}_{II}^{(2)^{-1}} \underline{\underline{A}}_{I\Gamma}^{(2)} \right) \underline{u}_\Gamma^k = \underline{f}_\Gamma - \underline{\underline{A}}_{\Gamma I}^{(2)} \underline{\underline{A}}_{II}^{(2)^{-1}} \underline{f}_I^{(2)} - \hat{\underline{\underline{A}}}_{\Gamma I}^{(1)} \underline{u}_I^{(1)^{k-1}} -$ [C]

$- \hat{\underline{\underline{A}}}_{\Gamma\Gamma}^{(1)} \underline{u}_\Gamma^{k-1}$

where $\hat{\underline{\underline{A}}}^{(i)} = \underline{\underline{A}}^{(i)} - \gamma_i \underline{\underline{M}}^{(i)}$

Imposing the first equation [A] in [C]

$$\left(\hat{\underline{\underline{A}}}_{\Gamma\Gamma}^{(2)} - \hat{\underline{\underline{A}}}_{\Gamma I}^{(2)} \underline{\underline{A}}_{II}^{(2)^{-1}} \underline{\underline{A}}_{I\Gamma}^{(2)} \right) \underline{u}_\Gamma^k = \underline{f}_\Gamma - \underline{\underline{A}}_{\Gamma I}^{(2)} \underline{\underline{A}}_{II}^{(2)^{-1}} \underline{f}_I^{(2)} - \hat{\underline{\underline{A}}}_{\Gamma I}^{(1)} \underline{\underline{A}}_{II}^{(1)^{-1}} \left(\underline{f}_I^{(1)} - \underline{\underline{A}}_{I\Gamma}^{(1)} \underline{u}_\Gamma^{\overset{k-1}{\uparrow}\ell} \right) -$$

[D] $$- \hat{\underline{\underline{A}}}_{\Gamma\Gamma}^{(1)} \underline{u}_\Gamma^{k-1}$$

$$\Rightarrow \left(\hat{\underline{\underline{A}}}_{\Gamma\Gamma}^{(2)} - \hat{\underline{\underline{A}}}_{\Gamma I}^{(2)} \underline{\underline{A}}_{II}^{(2)^{-1}} \underline{\underline{A}}_{I\Gamma}^{(2)} \right) \underline{u}_\Gamma^k = \underline{f}_\Gamma - \underline{\underline{A}}_{\Gamma I}^{(2)} \underline{\underline{A}}_{II}^{(2)^{-1}} \underline{f}_I^{(2)} - \hat{\underline{\underline{A}}}_{\Gamma I}^{(1)} \underline{\underline{A}}_{II}^{(1)^{-1}} \underline{f}_I^{(1)} - \left(\hat{\underline{\underline{A}}}_{\Gamma\Gamma}^{(1)} - \hat{\underline{\underline{A}}}_{\Gamma I}^{(1)} \underline{\underline{A}}_{II}^{(1)^{-1}} \underline{\underline{A}}_{I\Gamma}^{(1)} \right) \underline{u}_\Gamma^{k-1}$$

$\Rightarrow$ bringing everything to the same time step "k"

$$\left(\hat{\underline{\underline{A}}}_{\Gamma\Gamma}^{(2)} - \hat{\underline{\underline{A}}}_{\Gamma I}^{(2)} \underline{\underline{A}}_{II}^{(2)^{-1}} \underline{\underline{A}}_{I\Gamma}^{(2)} + \hat{\underline{\underline{A}}}_{\Gamma\Gamma}^{(1)} - \hat{\underline{\underline{A}}}_{\Gamma I}^{(1)} \underline{\underline{A}}_{II}^{(1)^{-1}} \underline{\underline{A}}_{I\Gamma}^{(1)} \right) \underline{u}^k = \underline{F}_\Gamma - \hat{\underline{\underline{A}}}_{\Gamma I}^{(2)} \underline{\underline{A}}_{II}^{(2)^{-1}} \underline{f}_I^{(2)} - \hat{\underline{\underline{A}}}_{\Gamma I}^{(1)} \underline{\underline{A}}_{II}^{(1)^{-1}} \underline{f}_I^{(1)}$$

that can be written as $\underline{\underline{S}} \underline{u}_\Gamma = \underline{G}$, which is the discrete/matrix version of the Steklov-Poincaré operator S

Question (d)

We can call
$$\underline{\underline{S}}_1 = \hat{\underline{\underline{A}}}_{\Gamma\Gamma}^{(1)} - \hat{\underline{\underline{A}}}_{\Gamma I}^{(1)} \underline{\underline{A}}_{II}^{(1)^{-1}} \underline{\underline{A}}_{I\Gamma}^{(1)}$$
$$\underline{\underline{S}}_2 = \hat{\underline{\underline{A}}}_{\Gamma\Gamma}^{(2)} - \hat{\underline{\underline{A}}}_{\Gamma I}^{(2)} \underline{\underline{A}}_{II}^{(2)^{-1}} \underline{\underline{A}}_{I\Gamma}^{(2)}$$
$$\underline{G} = \underline{f}_\Gamma - \hat{\underline{\underline{A}}}_{\Gamma I}^{(2)} \underline{\underline{A}}_{II}^{(2)^{-1}} \underline{f}_I^{(2)} - \hat{\underline{\underline{A}}}_{\Gamma I}^{(1)} \underline{\underline{A}}_{II}^{(1)^{-1}} \underline{f}_I^{(1)}$$

recalling the equation [D] of the previous question:

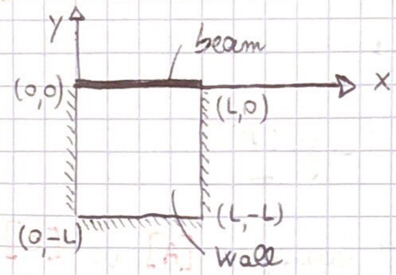
$$\underline{\underline{S}}_2\, \underline{u}_\Gamma^{(k)} = \underline{G} - \underline{\underline{S}}_1\, \underline{u}_\Gamma^{(k-1)} = \underline{G} - \underline{\underline{S}}\, \underline{u}_\Gamma^{k-1} + \underline{\underline{S}}_2\, \underline{u}_\Gamma^{k-1}$$

$$\Rightarrow \quad \underline{u}_\Gamma^{(k)} = \underline{\underline{S}}_2^{-1}\left(\underline{G} - \underline{\underline{S}}\, \underline{u}_\Gamma^{k-1}\right) + \underline{u}_\Gamma^{(k-1)} \qquad \underline{\underline{S}} = \underline{\underline{S}}_1 + \underline{\underline{S}}_2$$

So the Richardson preconditioner is $\underline{\underline{P}} = \underline{\underline{S}}_2^{-1}$

Exercise 3 : COUPLING OF HETEROGENEOUS PROBLEMS

Problem 1 : BEAM - WALL



Question (a)

The wall is modeled as flat plane, everything symmetric, uniform displacements inplane along the thickness, transverse stresses are negligible. $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$ PLANE STRESS HYPOTHESIS

A] Constitutive Law

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\gamma_{xy} \end{bmatrix}$$

B] **Displacements / Strains** (Kinematics)

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \end{bmatrix}$$

So that the stresses show the following structure:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} \frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} \\ \nu \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\ \frac{1-\nu}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \end{bmatrix}$$

C] Equilibrium

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{b} = \underline{0} \quad , \quad \underline{b} := \text{body forces}$$

$$\underline{\underline{\sigma}} := \text{stress tensor} : \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}$$

final wall equations:

$$\frac{E}{1-\nu^2}\begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial y}\frac{1-\nu}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ \frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial x}\frac{1-\nu}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\end{bmatrix}+\begin{bmatrix} b_x \\ b_y\end{bmatrix}=\underline{0}$$

with particular Boundary Conditions:

Displacements fixed on the lateral and bottom sides, the traction forces of the beam is the B.C. on the top of the wall

Question (b)

Since the wall gives to the beam a distributed transversal load q, the coupling that goes inside the beam equation comes from the stresses (normal) in y-direction on the top side of the wall ($\sigma_{yy}|_{y=0}$). Since it must be reconducted to a line load, the thickness "t" must be taken into count. So the modified beam equation will be like:

$$EI\frac{d^4 v}{dx^4}=f(x)-t\cdot\sigma_{yy}(x,y)\Big|_{y=0}$$

the "−" is given by the countrary direction to the f which goes like "−y" (it works like a reaction to the beam)

with $\sigma_{yy}=\frac{E}{1-\nu^2}\left(\nu\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\Big|_{y=0}$

Question (c)

Given Γ as the interface between the wall and the beam, represented by the straight line y=0, the vertical displacement "v" must be the same in the wall and in the beam. This means that its jump across the interface Γ must be equal to zero.

$$[\![v]\!]_\Gamma=0$$

Imposing the fact that the integral in the variational formulation must be additive, we obtain that the forces traction $\underline{t}$ normal to the top side of the wall must be the same in the two elements (beam, wall). This means that:

$$[\![\underline{n} \cdot (\underline{\nabla} \cdot \underline{\underline{\sigma}})]\!]_\Gamma = \emptyset$$

Which is like to imposing the reaction of the wall must be equal to the load acting on the beam.

Question (d)

- Firstly, if we consider that the horizontal displacements "u" is the same in both beam and wall at $y=0$,
  $[\![u]\!] = \emptyset$ and the tangential component of the tractions the same too, $[\![\underline{t} \cdot (\underline{\nabla} \cdot \underline{\underline{\sigma}})]\!]_\Gamma = \emptyset$, then the whole Euler-Bernoulli theory wouldn't be anymore satisfied, because in this case x-components must get inside of the beam equation, but they are not.

- Since, the horizontal displacements and tractions in the beam are not calculable, a jump in the contact with the more complete wall problem must be taken into count.
  $\left([\![u]\!]_\Gamma \neq \emptyset \;,\; [\![\underline{t} \cdot (\underline{\nabla} \cdot \underline{\underline{\sigma}})]\!]_\Gamma \neq \emptyset \right)$

Problem 2 : DIRICHLET-TO-NEUMANN

S_D := DIRICHLET-TO-NEUMANN operator for the Darcy problem

S_S := " " " operator for the Stokes problem

The Steklov-Poincaré problem is the following

$$S_D(\lambda) = S_S(\lambda)$$

Where λ is the normal velocity on Γ, and Γ is the interface between the Darcy and the Stokes problem regions.

The equations governing the whole problem in a strong form are:

Stokes:
$$\begin{cases} -\nu \Delta \underline{u}_s + \underline{\nabla} p_s = \underline{f} \\ \underline{\nabla} \cdot \underline{u}_s = 0 \end{cases} \quad \text{in } \Omega_s$$

Darcy:
$$\begin{cases} \underline{u}_d + \underline{\underline{K}}\, \underline{\nabla} \varphi = \underline{0} \\ \underline{\nabla} \cdot \underline{u}_d = 0 \end{cases} \quad \text{in } \Omega_d$$

$\Downarrow$

the second problem can be simplified to the following:

$$-\underline{\nabla} \cdot \left(\underline{\underline{K}}\, \underline{\nabla} \varphi \right) = 0 \quad \text{in } \Omega_d$$

the main question now is the knowledge of the transmission conditions:

1) Continuity of the normal components of the velocities (due to the incompressibility of the fluid): $\underline{u}_s \cdot \underline{n} = \underline{u}_d \cdot \underline{n}$ on Γ

2) Continuity of normal stresses:
$$\varphi = p_s - \nu (\underline{n} \cdot \underline{\nabla} \underline{u}_s \cdot \underline{n}) \quad \text{on } \Gamma$$

3) Condition for the tangential component of the velocity [Beavers-Joseph]
$$\nu\, \underline{t} \cdot (\underline{n} \cdot \underline{\nabla} \underline{u}_s) = - \frac{\alpha_{BJ}}{\sqrt{\underline{\underline{K}}}} (\underline{u}_s - \underline{u}_d) \cdot \underline{t} \quad \text{on } \Gamma$$

the weak form of the Stokes' problem is the following

$$\int_{\Omega_s} \mu (\underline{\nabla} \delta \underline{u}_s) : (\underline{\nabla} \underline{u}_s)\, d\Omega - \int_{\Omega_s} p_s (\underline{\nabla} \cdot \delta \underline{u}_s)\, d\Omega - \int_{\cancel{\partial\Omega_s}\,\Gamma} \delta \underline{u}_s \cdot \left[\underline{n} \left(-p_s \underline{\underline{I}} + \nu \underline{\nabla}^s \underline{u}_s \right) \right] dS =$$
$$= \int_{\Omega_s} \delta \underline{u}_s \cdot \underline{f}\, d\Omega$$

and $-\int_{\Omega} q_s(\underline{\nabla}\cdot\underline{u}_s)\,d\Omega$, q_s is the test function for the continuity equation:

In other compact way, the same problem can be stated as follows:

$$\begin{cases} (\nu\underline{\nabla}\underline{u}_s, \underline{\nabla}\delta\underline{u}_s) - (p_s, \underline{\nabla}\cdot\delta\underline{u}_s) - (\delta\underline{u}_s, (-p_s\underline{\underline{I}} + \nu\underline{\nabla}^s\underline{u}_s)\cdot\underline{n})_\Gamma = (\delta\underline{u}_s, \underline{f}) \\ -(q_s, \underline{\nabla}\cdot\underline{u}_s) = 0 \end{cases}$$

decomposing the velocity at the interface:

$$\begin{cases} (\nu\underline{\nabla}\underline{u}_s, \underline{\nabla}\delta\underline{u}_s) - (p_s, \underline{\nabla}\cdot\delta\underline{u}_s) - (\delta u_s^n, \underline{n}\cdot(-p_s\underline{\underline{I}} + \nu\underline{\nabla}^s\underline{u}_s)\cdot\underline{n})_\Gamma - \\ -(\delta u_s^t, \underline{t}\cdot(-p_s\underline{\underline{I}} + \nu\underline{\nabla}^s\underline{u}_s)\cdot\underline{n})_\Gamma = (\delta\underline{u}_s, \underline{f}) \\ -(q_s, \underline{\nabla}\cdot\underline{u}_s) = 0 \end{cases}$$

Using the transmission conditions:

$$* \begin{cases} (\nu\underline{\nabla}\underline{u}_s, \underline{\nabla}\delta\underline{u}_s) - (p_s, \underline{\nabla}\cdot\delta\underline{u}_s) - (\delta u_s^n, \varphi)_\Gamma + (\delta u_s^t, \frac{\alpha_{BJ}}{\sqrt{\underline{\underline{K}}}}(\underline{u}_s - \underline{u}_d)\cdot\underline{t})_\Gamma = (\delta\underline{u}_s, \underline{f}) \\ -(q_s, \underline{\nabla}\cdot\underline{u}_s) = 0 \end{cases}$$

the weak form of the Darcy problem is the following:

$$\int_{\Omega_d} \underline{\nabla}q_d\cdot(\underline{\underline{K}}\,\underline{\nabla}\varphi)\,d\Omega - \int_{\partial\Omega_{d\Gamma}} q_d(\underline{u}_s\cdot\underline{n})\,dS = 0$$

q_d is the test function for Darcy

here, the first t.c. has been used

In compact way:

$$** \quad (\underline{\nabla}q_d, \underline{\underline{K}}\,\underline{\nabla}\varphi) - (\underline{u}_s\cdot\underline{n}, q_d)_\Gamma = 0$$

This way the velocity in the Darcy subdomain "disappears". Once discretized, the following quantities are defined:

1) $\underline{U} = \sum_{j=1}^{N_s} u_{s_j}\delta u_{s_j}$ (velocity unknowns) N_s: total number of nodes dofs in Stokes sdb (VEL.)

2) $\underline{P}_s = \sum_{j=1}^{N_q} p_{s_j} q_{s_j}$ (pressure in Stokes' subdomain) N_q: total numbers of dofs for pressure in Stokes

3) $\underline{\Phi} = \sum_{j=1}^{N_d} \varphi_j q_{d_j}$ (potential in Darcy's subdomain) N_d: total number of dofs for potential in Darcy

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4) $\underline{\underline{A}}_s = a_s(\underline{u}_s, \delta\underline{u}_s) = (\nu \underline{\nabla} \underline{u}_s, \underline{\nabla} \delta \underline{u}_s)$ Stiffness matrix for Stokes

5) $\underline{\underline{B}}^T = b(\delta u_s, p_s) = (p_s, \underline{\nabla} \cdot \delta \underline{u}_s)$ Divergence matrix

6) $\underline{\underline{B}} = b(u_s, q_s) = (q_s, \underline{\nabla} \cdot \underline{u}_s)$ //

7) $\underline{\underline{M}}_{\Gamma\Gamma} = m_\Gamma(\delta u^n_{s_j}, \varphi) = (\delta u^n_s, \varphi)_\Gamma$ Mass matrix for the interface modes

8) $\underline{\underline{A}}_D = a_d(\underline{\underline{K}} \underline{\nabla} \varphi, \underline{\nabla} q_d) = (\underline{\underline{K}} \underline{\nabla} \varphi, \underline{\nabla} q_d)$ Stiffness matrix for Darcy

9) $\underline{\underline{M}}^T_{\Gamma\Gamma} = m_\Gamma(u^n_s, q_d) = (\underline{u}^n_s \cdot \underline{n}, q_d)_\Gamma$ //

- Stokes Equation, "internal" modes: N_{sI} : number of interal nodes Stokes
  (for $\delta u_{s_i}, i = 1 \dots N_{sI}$) $N_{s\Gamma}$: number of interface nodes Stokes

$$\sum_{j=1}^{N_{sI}} a_s(\delta u_{s_j}, \delta u_{s_i}) u_{s_j} + \sum_{j=1}^{N_\Gamma} \sum_{k=1}^{d-1} a_s(\delta u^\Gamma_{s_i} \cdot t_k, \delta u_{s_i})(u_{s_j} \cdot t_k) +$$

$$+ \sum_{j=1}^{N_s^\Gamma} a_s(\delta u^\Gamma_{s_i} \cdot n, \delta u_{s_i})(u_{s_j} \cdot n) - \sum_{j=1}^{N_q} b(\delta u_{s_i}, q_{s_j}) p_{s_j} = (\underline{f}, \delta u_{s_i})$$

$\underline{U}_I$ is the sum of the internal modes for the Stokes subdomain and the dof for the tangential velocity on the interface. $\underline{U}_\Gamma$ will be, instead, the normal velocity on the interface (we will call it for simplicity $\underline{\lambda}$).

$$\underline{\underline{A}}_{s_{II}} \underline{U}_I + \underline{\underline{A}}_{s_{I\Gamma}} \underline{\lambda} - \underline{\underline{B}}^T \underline{P}_s = \underline{f}_I$$

- - Stokes Equation, λ dofs:
  for $\delta u^\Gamma_{s_i}, i = 1 \dots N_{s\Gamma}$

$$\sum_{j=1}^{N_I} a_s(\delta u_{s_j}, \delta u^\Gamma_{s_i}) u_{s_j} + \sum_{j=1}^{N^s_\Gamma} \sum_{k=1}^{d-1} a_s(\delta u^\Gamma_{s_i} \cdot t_k, \delta u^\Gamma_{s_i})(u_{s_j} \cdot t_k) +$$

$$+\sum_{j=1}^{N_\Gamma} a_s(\delta u^\Gamma_{s_i}\cdot n, \delta u^\Gamma_{s_i})(u_{s_j}\cdot n) + \sum_{j=1}^{N^s_\Gamma}\sum_{k=1}^{d-1}(\delta u^\Gamma_{s_i}\cdot t_k, \frac{\alpha_{BJ}}{\sqrt{K}}\delta u^\Gamma_{s_j}\cdot t_k)\cdot$$

$$\cdot u_{s_j} - \sum_{j=1}^{N_q} b(\delta u^\Gamma_{s_i}, q_{s_j})p_{s_j} - \sum_{j=1}^{N^s_\Gamma}(q_{d_j}, \delta u^\Gamma_{s_i}\cdot n) = (f, \delta u^\Gamma_{s_i})$$

$$\Downarrow$$

$$\underline{\underline{A}}_{s_{\Gamma I}}\underline{U}_I + \underline{\underline{A}}_{s_{\Gamma\Gamma}}\underline{\lambda} - \underbrace{\underline{\underline{B}}^T_{I\Gamma}}_{\underline{\underline{B}}_{\Gamma I}}\underline{P}_s - \underline{\underline{M}}_{\Gamma\Gamma}\underline{\Phi}_\Gamma = \underline{f}_\Gamma$$

••• Darcy Equation, internal nodes:

for q_{d_i}, $i = 1 \dots\dots N_{d_I}$

N_{d_I}: internal nodes Darcy

N_{d_Γ}: interface nodes Darcy

$$\sum_{j=1}^{N_{d_I}} a_d(q_{d_j}, q_{d_i})\varphi_j + \sum_{j=1}^{N_{d_\Gamma}} a_d(q^\Gamma_{d_j}, q_{d_i})\varphi_j = 0$$

$$\underline{\underline{A}}_{D_{II}}\underline{\Phi}_I + \underline{\underline{A}}_{D_{I\Gamma}}\underline{\Phi}_\Gamma = \underline{0}$$

•••• Darcy Equation, interface nodes:

for q_{d_i}, $i = 1 \dots\dots N_{d_\Gamma}$

$$\sum_{j=1}^{N_{d_I}} a_d(q_{d_j}, q_{d_i})\varphi_j + \sum_{j=1}^{N_{d_\Gamma}} a_d(q^\Gamma_{d_j}, q^\Gamma_{d_i})\varphi_j - \sum_{j=1}^{N_{d_\Gamma}}(q^\Gamma_{d_i}, \delta u^\Gamma_{s_j}\cdot n)\cdot$$

$$\cdot(u_{s_j}\cdot n) = 0$$

$$\underline{\underline{A}}_{D_{\Gamma I}}\underline{\Phi}_I + \underline{\underline{A}}_{D_{\Gamma\Gamma}}\underline{\Phi}_\Gamma - \underline{\underline{M}}^T_{\Gamma\Gamma}\underline{\lambda} = \underline{0}$$

••••• Incompressibility condition of the Stokes problem:

for q_{s_i}, $i = 1 \dots\dots N_q$:

$$-\sum_{j=1}^{N_{s_I}} b(\delta u_{s_j}, q_{s_i})u_{s_j} + \sum_{j=1}^{N_{s_\Gamma}} -b(\delta u^\Gamma_{s_j}\cdot n, q_{s_i})(u^\Gamma_{s_j}\cdot n) +$$

$$+\sum_{j=1}^{N_{s_\Gamma}}\sum_{k=1}^{d-1} -b(\delta u^\Gamma_{s_j}\cdot t_k, q_{s_i})(u^\Gamma_{s_j}\cdot t_k) = 0$$

$$-\underline{\underline{B}}_{II}\underline{U}_I - \underline{\underline{B}}_{I\Gamma}\underline{\lambda} = \underline{0}$$ (here, again, the $\underline{U}_I$ includes the tangential velocity on the interface)

The global problem can be sintetited as follows:

$$\begin{bmatrix} \underline{\underline{A}}_{s_{II}} & -\underline{\underline{B}}^T & \underline{\underline{A}}_{s_{I\Gamma}} & \underline{\underline{0}} & \underline{\underline{0}} \\ -\underline{\underline{B}}_{II} & \underline{\underline{0}} & -\underline{\underline{B}}_{I\Gamma} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{A}}_{s_{\Gamma I}} & -\underline{\underline{B}}_{\Gamma I} & \underline{\underline{A}}_{s_{\Gamma\Gamma}} & -\underline{\underline{M}}_{\Gamma\Gamma} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & -\underline{\underline{M}}^T_{\Gamma\Gamma} & \underline{\underline{A}}_{d_{\Gamma\Gamma}} & \underline{\underline{A}}_{d_{\Gamma I}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{A}}_{d_{I\Gamma}} & \underline{\underline{A}}_{d_{II}} \end{bmatrix} \begin{bmatrix} \underline{U}_I \\ \underline{P}_s \\ \underline{\lambda} \\ \underline{\varphi}_\Gamma \\ \underline{\varphi}_I \end{bmatrix} = \begin{bmatrix} \underline{f}_I \\ \underline{0} \\ \underline{f}_\Gamma \\ \underline{0} \\ \underline{0} \end{bmatrix}$$

which represent a Steklov-Pointcaré equation

Question (b)

$$\begin{bmatrix} \underline{\underline{A}}_{d_{\Gamma\Gamma}} & \underline{\underline{A}}_{d_{\Gamma I}} \\ \underline{\underline{A}}_{d_{I\Gamma}} & \underline{\underline{A}}_{d_{II}} \end{bmatrix} \begin{bmatrix} \underline{\varphi}_\Gamma \\ \underline{\varphi}_I \end{bmatrix}^{k+1} = \begin{bmatrix} \underline{\underline{M}}^T_{\Gamma\Gamma} \underline{\lambda}^k \\ \underline{0} \end{bmatrix}$$

firstly we solve the Darcy Problem, given the normal velocity on the Interface. Then, using the potentiole on the interface $\underline{\varphi}_\Gamma$, we solve the Stokes problem

$$\begin{bmatrix} \underline{\underline{A}}_{s_{II}} & -\underline{\underline{B}}^T & \underline{\underline{A}}_{s_{I\Gamma}} \\ -\underline{\underline{B}}_{II} & \underline{\underline{0}} & -\underline{\underline{B}}_{I\Gamma} \\ \underline{\underline{A}}_{s_{\Gamma I}} & -\underline{\underline{B}}_{\Gamma I} & \underline{\underline{A}}_{s_{\Gamma\Gamma}} \end{bmatrix} \begin{bmatrix} \underline{U}_I \\ \underline{P}_s \\ \underline{\lambda} \end{bmatrix}^{k+1} = \begin{bmatrix} \underline{f}_I \\ \underline{0} \\ \underline{f}_\Gamma + \underline{\underline{M}}_{\Gamma\Gamma} \underline{\varphi}_\Gamma^{l} \end{bmatrix}$$

l can be $\begin{cases} k+1 & \text{Gauss Seidel} \\ k & \text{Jacobi} \end{cases}$

Question (c)

Let's call the previous global system as follows: $\underline{\underline{\tilde{A}}}\,\underline{U} = \underline{F}$

The Richardson iterative schemes says that: $\underline{U}^{k+1} = \underline{U}^k + \underline{\underline{\tilde{B}}}^{-1} \underline{R}^k$

where $\underline{R}^k$ is the residual at the iteration "k": $\underline{R}^k = \underline{F} - \tilde{\underline{\underline{A}}}\,\underline{U}^k$

Let us use a Jacobi - Additive Scheme: So we will have the following problems:

1) Darcy:

$$\begin{bmatrix} \underline{\underline{A}}_{d_{\Gamma\Gamma}} & \underline{\underline{A}}_{d_{\Gamma I}} \\ \underline{\underline{A}}_{d_{I\Gamma}} & \underline{\underline{A}}_{d_{II}} \end{bmatrix} \begin{bmatrix} \underline{\varphi}_\Gamma^{k+1} \\ \underline{\varphi}_I^{k+1} \end{bmatrix} = \begin{bmatrix} \underline{\underline{M}}_{\Gamma\Gamma}^T \underline{\lambda}^k \\ \underline{0} \end{bmatrix}$$

2) Stokes:

$$\begin{bmatrix} \underline{\underline{A}}_{s_{II}} & -\underline{\underline{B}}^T & \underline{\underline{A}}_{s_{I\Gamma}} \\ -\underline{\underline{B}}_{II} & \underline{\underline{0}} & -\underline{\underline{B}}_{I\Gamma} \\ \underline{\underline{A}}_{s_{\Gamma I}} & -\underline{\underline{B}}_{\Gamma I} & \underline{\underline{A}}_{s_{\Gamma\Gamma}} \end{bmatrix} \begin{bmatrix} \underline{U}_I^{k+1} \\ \underline{P}_s^{k+1} \\ \underline{\lambda}^{k+1} \end{bmatrix} = \begin{bmatrix} \underline{f}_I \\ \underline{0} \\ \underline{f}_\Gamma + \underline{\underline{M}}_{\Gamma\Gamma}\underline{\varphi}_\Gamma^k \end{bmatrix}$$

Which can be assembled as follows:

$$\begin{bmatrix} \underline{\underline{A}}_{s_{II}} & -\underline{\underline{B}}^T & \underline{\underline{A}}_{s_{I\Gamma}} & \underline{\underline{0}} & \underline{\underline{0}} \\ -\underline{\underline{B}}_{II} & \underline{\underline{0}} & -\underline{\underline{B}}_{I\Gamma} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{A}}_{s_{\Gamma I}} & -\underline{\underline{B}}_{\Gamma I} & \underline{\underline{A}}_{s_{\Gamma\Gamma}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{A}}_{d_{\Gamma\Gamma}} & \underline{\underline{A}}_{d_{\Gamma I}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{A}}_{d_{I\Gamma}} & \underline{\underline{A}}_{d_{II}} \end{bmatrix} \begin{bmatrix} \underline{U}_I \\ \underline{P}_s \\ \underline{\lambda} \\ \underline{\varphi}_\Gamma \\ \underline{\varphi}_I \end{bmatrix}^{k+1} = \begin{bmatrix} \underline{f}_I \\ \underline{0} \\ \underline{f}_\Gamma \\ \underline{0} \\ \underline{0} \end{bmatrix} + \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & -\underline{\underline{M}}_{\Gamma\Gamma}^T & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & -\underline{\underline{M}}_{\Gamma\Gamma} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{U}_I \\ \underline{P}_s \\ \underline{\lambda} \\ \underline{\varphi}_\Gamma \\ \underline{\varphi}_I \end{bmatrix}^{k}$$

in short way: $\tilde{\underline{\underline{B}}}\,\underline{U}^{k+1} = \underline{F} + \tilde{\underline{\underline{E}}}\,\underline{U}^k$

where $\tilde{\underline{\underline{E}}}$ is called the perturbation matrix such that $\tilde{\underline{\underline{B}}} = \tilde{\underline{\underline{A}}} + \tilde{\underline{\underline{E}}}$

Given a first guess $\underline{U}^0$, compute for $k = 0, \ldots$

the following iteration: $\tilde{\underline{\underline{B}}}\,\underline{U}^{k+1} = \underline{F} + \tilde{\underline{\underline{E}}}\,\underline{U}^k$

If $\{\underline{U}^k\}$ converges to $\underline{U}$, it can be written as

$$\underline{\underline{\tilde{B}}}\,\underline{U}^{k+1} = \underline{F} + \underline{\underline{\tilde{E}}}\,\underline{U}^k = \underline{F} - \underline{\underline{\tilde{A}}}\,\underline{U}^k + \underline{\underline{\tilde{B}}}\,\underline{U}^k = \underline{R}^k + \underline{\underline{\tilde{B}}}\,\underline{U}^k$$

Pre-multiplying by $\underline{\underline{\tilde{B}}}^{-1}$ both sides:

$$\underline{\underline{\tilde{B}}}^{-1}\,\underline{\underline{\tilde{B}}}\,\underline{U}^{k+1} = \underline{\underline{\tilde{B}}}^{-1}\left(\underline{R}^k + \underline{\underline{\tilde{B}}}\,\underline{U}^k\right)$$

Which brings to:

$$\boxed{\underline{U}^{k+1} = \underline{U}^k + \underline{\underline{\tilde{B}}}^{-1}\,\underline{R}^k}$$ which is the Richardson iteration scheme

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Exercise 4: MONOLITHIC AND PARTITIONED SCHEMES IN TIME

Consider the one-dimensional, transient, heat transfer equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial^2 u}{\partial x^2} = f & \text{in } [0,1] \\ u(x=0,t) = 0 \\ u(x=1,t) = 0 \\ u(x,t=0) = 0 \end{cases}$$

$\boxed{\text{Problem 1}}$

- Firstly let's do the temporal discretization: Firstly let's use the more general θ-method, then, imposing $\theta = 1$, we will show the Backward Euler scheme (BDF 1):
  ↓ stop at first order

$$\frac{u^{m+1} - u^m}{\Delta t} = \theta\left(\frac{\partial u}{\partial t}\right)^{m+1} + (1-\theta)\left(\frac{\partial u}{\partial t}\right)^m$$

Calling the following quantities:

$$\Delta u = u^{m+1} - u^m \qquad \Delta u_t = \left(\frac{\partial u}{\partial t}\right)^{m+1} - \left(\frac{\partial u}{\partial t}\right)^m$$

We have:

$$\frac{\Delta u}{\Delta t} - \theta\,\Delta u_t = u_t^m, \quad \text{where } u_t^m = \left(\frac{\partial u}{\partial t}\right)^m$$

Since $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} + f$, and calling $\Delta u_{xx} = \left(\dfrac{\partial^2 u}{\partial x^2}\right)^{m+1} - \left(\dfrac{\partial^2 u}{\partial x^2}\right)^m$

we obtain :

$$\frac{\Delta u}{\Delta t} - \theta k \, \Delta u_{xx} = \theta f^{n+1} + (1-\theta) f^{n} + k \, u_{xx}^{n}$$

Given the values for θ, k and $f = 1$ (constant)

$$\frac{\Delta u}{\Delta t} - \Delta u_{xx} = 1 + u_{xx}^{n}$$

•• Using the space finite element discretization, starting from using a test function δu such that $\delta u = 0$ on $x=0$ and $x=1$

$$\int_0^1 \delta u \frac{\Delta u}{\Delta t} dx - \int_0^1 \delta u \frac{d^2 \Delta u}{dx^2} dx = \int_0^1 \delta u \, dx + \int_0^1 \delta u \left(\frac{d^2 u}{dx^2}\right)^n dx =$$

$u, \delta u \in H_0^1([0,1])$

Imaging size h, such that

$$\underline{U} \cong \underline{U}_h = \sum_j u_j N_j(x)$$

$$\delta \underline{u} \cong \delta \underline{u}_h = \sum_i N_i(x)$$

$$= \int_0^1 \delta u \frac{\Delta u}{\Delta t} dx + \int_0^1 \frac{d \delta u}{dx} \frac{d \Delta u}{dx} dx - \delta u \left.\underbrace{\frac{d \Delta u}{dx}}_{0}\right|_{x=0/1} = \int_0^1 \delta u \, dx +$$

(given the B.C.)

$$+ \; \delta u \left.\underbrace{\left(\frac{du}{dx}\right)^n}_{0}\right|_{x=0/1} - \int_0^1 \frac{d \delta u}{dx} \left(\frac{d u}{dx}\right)^n dx$$

(given the B.C.)

We will have

$$\left(\frac{\Delta u}{\Delta t}, \delta u\right) + \left(\frac{d \Delta u}{dx}, \frac{d \delta u}{dx}\right) = (1, \delta u) - \left(\frac{du^n}{dx}, \frac{d \delta u}{dx}\right)$$

$\Delta t \to 1$

first derivative

$$\sum_j (N_i, N_j \Delta u_j)_\Omega + \sum_j (N_i', N_j' \Delta u_j)_\Omega = \sum_j (N_i, 1)_\Omega - \sum_j (N_i', N_j' u_j^n)_\Omega$$

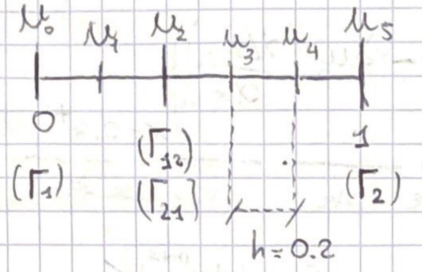
$$\underline{\underline{M}} \Delta \underline{U} + \underline{\underline{K}} \Delta \underline{U} = \underline{f_1} - \underline{\underline{K}} \, \underline{U}^n$$

knowing that $\Delta \underline{U} = \underline{U}^{m+1} - \underline{U}^{m}$

the final algebraic relation is : $\boxed{(\underline{\underline{M}} + \underline{\underline{K}})\, \underline{U}^{m+1} = \underline{f_1} + \underline{\underline{M}}\,\underline{U}^{m}}$

Problem 2

Using the following domain decomposition:



for subdomain 1, we have $\Omega_1 = [0, 0.4^-]$

boundary $\Gamma_1 \Rightarrow x = 0$

$\Gamma_{12} \Rightarrow x = 0.4^-$

Using the previous scheme we have :

$$\int_0^{0.4^-} \delta u_1 \frac{\Delta u_1}{\Delta t}\, dx + \int_0^{0.4^-} k \frac{d\delta u_1}{dx} \frac{d\Delta u_1}{dx}\, dx - k\, \delta u_1 \frac{d\Delta u_1}{dx}\Bigg|_{\cancel{x=0}}^{x=0.4^-} = \int_0^{0.4^-} \delta u_1 f\, dx +$$

(given b.c)

$$+ k\, \delta u_1 \left(\frac{du_1}{dx}\right)^m \Bigg|_{\cancel{x=0}}^{x=0.4^-} - \int_0^{0.4^-} k \frac{d\delta u_1}{dx} \left(\frac{du_1}{dx}\right)^m dx$$

(given b.c.)

for subdomain 2, we have $\Omega_2 = [0.4^+, 1]$

boundary $\Gamma_{21} \Rightarrow x = 0.4^+$

$\Gamma_2 \Rightarrow x = 1$

(given b.c.)

$$\int_{0.4^+}^{1} \delta u_2 \frac{\Delta u_2}{\Delta t}\, dx + \int_{0.4^+}^{1} k \frac{d\delta u_2}{dx} \frac{d\Delta u_2}{dx}\, dx - k\, \delta u_2 \frac{d\Delta u_2}{dx}\Bigg|_{x=0.4^+}^{\cancel{x=1}} = \int_{0.4^+}^{1} f\, \delta u_2\, dx +$$

$$+ k\, \delta u_2 \left(\frac{du_2}{dx}\right)^m \Bigg|_{x=0.4^+}^{\cancel{x=1}} \text{(given b.c.)} - \int_{0.4^+}^{1} k \frac{d\delta u_2}{dx} \left(\frac{du_2}{dx}\right)^m dx$$

Since the total mesh uses the same interpolation space ($\delta u =$
$= \delta u_1 = \delta u_2$) and the fact that $u = 0$ on $x=0$, $x=1$
we can use the property of additivity and the fact that

$$k\,\delta u_1 \left(\frac{du_1}{dx}\right)^n \Bigg|_{x=0.4^-} = -k\delta u_2 \left(\frac{du_2}{dx}\right)^n \Bigg|_{x=0.4^+}$$

$$k\,\delta u_1 \left(\frac{d\Delta u_1}{dx}\right) \Bigg|_{x=0.4^-} = -k\delta u_2 \left(\frac{d\Delta u_2}{dx}\right) \Bigg|_{x=0.4^+}$$

$\Downarrow$ simplifying the problem we have: (we use the same test δu)

$$\int_0^{0.4^-} \delta u \cdot \frac{u_1^{n+1}}{\Delta t}\,dx + \int_0^{0.4^-} k\frac{d\delta u}{dx}\frac{du_1^{n+1}}{dx}\,dx + \cancel{k\delta u\left(\frac{du_1^{n+1}}{dx}\right)\Bigg|_{x=0.4^-}} +$$

$$+ \int_{0.4^+}^{1} \delta u \cdot \frac{u_2^{n+1}}{\Delta t}\,dx + \int_{0.4^+}^{1} k\frac{d\delta u}{dx}\frac{du_2^{n+1}}{dx}\,dx - \cancel{k\delta u\left(\frac{du_2^{n+1}}{dx}\right)\Bigg|_{x=0.4^+}} =$$

$$= \int_0^{0.4^-} \delta u f\,dx + \int_0^{0.4^-} \delta u \cdot \frac{u_1^n}{\Delta t}\,dx + \int_{0.4^+}^{1} \delta u f\,dx + \int_{0.4^+}^{1} \delta u \cdot \frac{u_2^n}{\Delta t}\,dx$$

Using additivity property, this is equal to the global problem in which:

$$\int_0^1 \delta u \cdot u^{n+1}\,dx + \int_0^1 k\frac{d\delta u}{dx}\frac{du^{n+1}}{dx}\,dx = \int_0^1 f\delta u\,dx + \int_0^1 \delta u \cdot u^n\,dx$$

So there is no boundary integrals required at the interface (✓)

Problem 3

Let's use the following notation: $\underline{U}_I^{(1)} = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$, $\underline{U}_r = \{u_2\}$, $\underline{U}_I^{(2)} = \begin{bmatrix} u_3 \\ u_4 \\ u_5 \end{bmatrix}$

we need to solve for the left domain, knowing that

$\underline{U}^n$ is known and $\underline{U}_r^{n+1}$ is known as well

The solution is concentrating on $[\underline{U}_I^{(1)^{n+1}}]$.

The Global problem is written as follows:

$$\left(\frac{\underline{\underline{M}}}{\Delta t}+\underline{\underline{K}}\right)\underline{U}^{n+1}=\underline{f}+\frac{\underline{\underline{M}}}{\Delta t}\underline{U}^{n}$$

But since the problem is monolithic, the same equation is in term of the first subdomain only:

$$\underbrace{\left(\frac{\underline{\underline{M}}^{(1)}}{\Delta t}+\underline{\underline{K}}^{(1)}\right)}_{\underline{\underline{A}}^{(1)}}\underline{U}^{(1)^{n+1}}=\underline{f}^{(1)}+\frac{\underline{\underline{M}}^{(1)}}{\Delta t}\underline{U}^{n}_{(1)}$$

Can be described as follows: (Using $\Delta t = 1$ for simplicity)

$$\begin{bmatrix}\underline{\underline{A}}_{II}^{(1)} & \underline{\underline{A}}_{I\Gamma}^{(1)}\\ \underline{\underline{A}}_{\Gamma I}^{(1)} & \underline{\underline{A}}_{\Gamma\Gamma}^{(1)}\end{bmatrix}\begin{bmatrix}[\underline{U}_I^{(1)^{n+1}}]\\ \underline{U}_\Gamma^{n+1}\end{bmatrix}=\begin{bmatrix}\underline{f}_I^{(1)}\\ \underline{f}_\Gamma^{(1)}\end{bmatrix}+\begin{bmatrix}\underline{\underline{M}}_{II}^{(1)} & \underline{\underline{M}}_{I\Gamma}^{(1)}\\ \underline{\underline{M}}_{\Gamma I}^{(1)} & \underline{\underline{M}}_{\Gamma\Gamma}^{(1)}\end{bmatrix}\begin{bmatrix}\underline{U}_I^{(1)^n}\\ \underline{U}_\Gamma^{n}\end{bmatrix}$$

this value represents a Dirichlet boundary condition for the left subdomain

Extracting from the equation the only unknown ([ ]), we have

$$\underline{\underline{A}}_{II}^{(1)}\underline{U}_I^{(1)^{n+1}}+\underline{\underline{A}}_{I\Gamma}^{(1)}\underline{U}_\Gamma^{n+1}=\underline{f}_I^{(1)}+\underline{\underline{M}}_{II}^{(1)}\underline{U}_I^{(1)^n}+\underline{\underline{M}}_{I\Gamma}^{(1)}\underline{U}_\Gamma^{n}$$

$$\Downarrow$$

$$\underline{U}_I^{(1)^{n+1}}=\underline{\underline{A}}_{II}^{(1)^{-1}}\left(\underline{f}_I^{(1)}+\underline{\underline{M}}_{II}^{(1)}\underline{U}_I^{(1)^n}+\underline{\underline{M}}_{I\Gamma}^{(1)}\underline{U}_\Gamma^{n}-\underline{\underline{A}}_{I\Gamma}^{(1)}\underline{U}_\Gamma^{n+1}\right)$$

Problem 4

Given the fact that the monolithic ~~■~~ way of the problem makes us concentrate individually on the right sbd as an Individual Neumann problem we will have in a variational form:

$$\int_{0.4^+}^{1}\delta u\cdot\frac{u_2^{n+1}}{\Delta t}dx+\int_{0.4^+}^{1}k\frac{d\delta u}{dx}\frac{du_2}{dx}^{n+1}-k\delta u\left(\frac{du_2}{dx}^{n+1}\right)\Bigg|_{x=0.4^+}=$$

$$= \int_{0.4^+}^{1} \delta u\, f\, dx + \int_{0.4}^{1} \delta u \cdot \frac{u_2^n}{\Delta t}\, dx$$

Using a $\Delta t = 1$ for simplicity and the following relation

$$k\, \delta u \left.\frac{du_2^{n+1}}{dx}\right|_{x=0.4} = -k\, \delta u \left.\frac{du_1^{n+1}}{dx}\right|_{x=0.4}$$

$$\phi_2^{n+1} = -\phi_1^{n+1} = -\phi^{n+1}$$

↳ we focus on the quantity coming from the first slab

So we have:

$$(\underline{\underline{M}} + \underline{\underline{K}})\, \underline{U}^{n+1} = \underline{f} - \underline{\phi}^{n+1} + \underline{\underline{M}}\, \underline{U}^{n}$$

Where the $\underline{\phi}^{n+1}$ represents $\underline{\phi}^{n+1} = \begin{bmatrix} \underline{0}_I \\ \phi^{n+1} \end{bmatrix}$

Which means:

$$\begin{bmatrix} \underline{\underline{A}}_{II}^{(2)} & \underline{\underline{A}}_{I\Gamma}^{(2)} \\ \underline{\underline{A}}_{\Gamma I}^{(2)} & \underline{\underline{A}}_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} \underline{U}_I^{(2)\,n+1} \\ \underline{U}_\Gamma^{n+1} \end{bmatrix} = \begin{bmatrix} \underline{f}_I^{(2)} \\ \underline{f}_\Gamma^{(2)} \end{bmatrix} + \begin{bmatrix} \underline{0} \\ -\phi^{n+1} \end{bmatrix} + \begin{bmatrix} \underline{\underline{M}}_{II}^{(2)} & \underline{\underline{M}}_{I\Gamma}^{(2)} \\ \underline{\underline{M}}_{\Gamma I}^{(2)} & \underline{\underline{M}}_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} \underline{U}_I^{(2)\,n} \\ \underline{U}_\Gamma^{n} \end{bmatrix}$$

Since $\phi^{n+1} = \underline{\underline{A}}_{\Gamma\Gamma}^{(1)} \underline{U}_\Gamma^{n+1}$, and ◌ are the unknowns, we have

$$\underline{\underline{A}}_{\Gamma I}^{(2)} \underline{U}_I^{(2)\,n+1} + \underline{\underline{A}}_{\Gamma\Gamma}^{(2)} \underline{U}_\Gamma^{n+1} = \underline{f}_\Gamma^{(2)} - \underline{\underline{A}}_{\Gamma\Gamma}^{(1)} \underline{U}_\Gamma^{n+1} + \underline{\underline{M}}_{\Gamma I}^{(2)} \underline{U}_I^{(2)\,n} + \underline{\underline{M}}_{\Gamma\Gamma}^{(2)} \underline{U}_\Gamma^{n}$$

So $$\underline{U}_\Gamma^{n+1} = \underline{\underline{A}}_{\Gamma\Gamma}^{(2)\,-1} \left(\underline{f}_\Gamma^{(2)} - \underline{\underline{A}}_{\Gamma\Gamma}^{(1)} \underline{U}_\Gamma^{n+1} - \underline{\underline{A}}_{\Gamma I}^{(2)} \underline{U}_I^{(2)\,n+1} + \underline{\underline{M}}_{\Gamma I}^{(2)} \underline{U}_I^{(2)\,n} + \underline{\underline{M}}_{\Gamma\Gamma}^{(2)} \underline{U}_\Gamma^{n} \right)$$

putting in the first equation we have:

$$\underline{\underline{A}}_{II}^{(2)} \underline{U}_I^{(2)\,n+1} + \underline{\underline{A}}_{I\Gamma}^{(2)} \underline{\underline{A}}_{\Gamma\Gamma}^{(2)\,-1} \left(\underline{f}_\Gamma^{(2)} - \underline{\underline{A}}_{\Gamma\Gamma}^{(1)} \underline{U}_\Gamma^{n+1} - \underline{\underline{A}}_{\Gamma I}^{(2)} \underline{U}_I^{(2)\,n+1} + \underline{\underline{M}}_{\Gamma I}^{(2)} \underline{U}_I^{(2)\,n} + \underline{\underline{M}}_{\Gamma\Gamma}^{(2)} \underline{U}_\Gamma^{n} \right)$$

$$= \underline{f}_I^{(2)} + \underline{\underline{M}}_{II}^{(2)} \underline{U}_I^{(2)\,n} + \underline{\underline{M}}_{I\Gamma}^{(2)} \underline{U}_\Gamma^{n}$$

which is equal to:

$$\underline{U}_I^{(2)^{m+1}} = \left(\underline{\underline{A}}_{II}^{(2)} - \underline{\underline{A}}_{I\Gamma}^{(2)} \underline{\underline{A}}_{\Gamma\Gamma}^{(2)^{-1}} \underline{\underline{A}}_{\Gamma I}^{(2)}\right)^{-1} \left(\underline{f}_I^{(2)} + \underline{\underline{M}}_{II}^{(2)} \underline{U}_I^{(2)^m} + \underline{\underline{M}}_{I\Gamma}^{(2)} \underline{U}_\Gamma^{m} - \underline{\underline{A}}_{I\Gamma}^{(2)} \underline{\underline{A}}_{\Gamma\Gamma}^{(2)^{-1}} \cdot \right.$$

$$\left. \cdot \left(\underline{f}_\Gamma^{(2)} - \underline{\underline{A}}_{\Gamma\Gamma}^{(1)} \underline{U}_\Gamma^{m+1} + \underline{\underline{M}}_{\Gamma I}^{(2)} \underline{U}_I^{(2)^m} + \underline{\underline{M}}_{\Gamma\Gamma}^{(2)} \underline{U}_\Gamma^{m}\right)\right)$$

Firstly we computing this equation, than the previous, which contains the just computed value for the interior nodes $\underline{U}_I^{(2)^{m+1}}$

Problem 5

The iterative algorithm for a staggered aproch is shown as follows:

- In the first (left) subdomain, in continuous form:

$$\begin{cases} \dfrac{\partial u^{i+1\,(1)}}{\partial t} - k \dfrac{\partial^2 u^{(1)\,i+1}}{\partial x^2} = f^{(1)} \\ u^{i+1\,(1)} = u^{i(2)} \quad \text{on } \Gamma_{12} \longrightarrow \begin{bmatrix}\text{DIRICHLET} \\ \text{PROBLEM}\end{bmatrix} \\ u^{(1)} = 0 \quad \text{on } \Gamma_1 \ (x=0) \end{cases}$$

Which brings to the following algebraic version:

$$\underline{U}_I^{(1)^{i+1,m+1}} = \underline{\underline{A}}_{II}^{(1)^{-1}} \left(\underline{f}_I^{(1)} + \underline{\underline{M}}_{II}^{(1)} \underline{U}_I^{(1)^m} + \underline{\underline{M}}_{I\Gamma}^{(1)} \underline{U}_\Gamma^{m} - \underline{\underline{A}}_{I\Gamma}^{(1)} \underline{U}_\Gamma^{m+1,i}\right)$$

- In the second (right) subdomain, in continuous form:

$$\begin{cases} \dfrac{\partial u^{i+1\,(2)}}{\partial t} - k \dfrac{\partial^2 u^{(2)\,i+1}}{\partial x^2} = f^{(2)} \\ k \dfrac{\partial u^{(2)\,i+1}}{\partial x} = -k \dfrac{\partial u^{(1)\,i}}{\partial x} \quad \text{on } \Gamma_{21} \longrightarrow \begin{bmatrix}\text{NEUMANN} \\ \text{PROBLEM}\end{bmatrix} \\ u^{(2)} = 0 \quad \text{on } \Gamma_2 \ (x=1) \end{cases}$$

Which brings to the following algebraic version:

$$\begin{bmatrix} \underline{\underline{A}}_{II}^{(2)} & \underline{\underline{A}}_{I\Gamma}^{(2)} \\ \underline{\underline{A}}_{\Gamma I}^{(2)} & \underline{\underline{A}}_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} \underline{U}_I^{(2)^{i+1,m+1}} \\ \underline{U}_\Gamma^{i+1,m+1} \end{bmatrix} = \begin{bmatrix} \underline{f}_I^{(2)} \\ \underline{f}_\Gamma^{(2)} - \Phi^{i,m+1} \end{bmatrix} + \begin{bmatrix} \underline{\underline{M}}_{II}^{(2)} & \underline{\underline{M}}_{I\Gamma}^{(2)} \\ \underline{\underline{M}}_{\Gamma I}^{(2)} & \underline{\underline{M}}_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} \underline{U}_I^{(2)^m} \\ \underline{U}_\Gamma^{m} \end{bmatrix}$$

with, $\Phi^{i,m+1} = \underline{\underline{A}}_{\Gamma\Gamma}^{(1)} \underline{U}_\Gamma^{i,m+1}$

with this scheme now each subdomain problem can be solved in a parallel scheme for the same time step.

Problem 6

The substitution scheme is the following:

- For the first (left) subdomain, we have the same of before:

$$\begin{cases} \dfrac{\partial u}{\partial t}^{i+1^{(1)}} - k \dfrac{\partial^2 u}{\partial x^2}^{i+1^{(1)}} = f^{(1)} \\ u^{(1)^{i+1}} = u^{(2)^{i}} \quad \text{on} \quad \Gamma_{12} \; (x = 0.4^-) \\ u^{(1)} = 0 \quad \text{on } x = 0 \end{cases} \longrightarrow \begin{bmatrix} \text{DIRICHLET} \\ \text{PROBLEM} \end{bmatrix}$$

which brings to the following algebraic version:

$$\underline{U}_{I}^{m+1, i+1}{}_{(1)} = \underline{\underline{A}}_{II}^{(1)^{-1}} \left(\underline{f}_I^{(1)} + \underline{\underline{M}}_{II}^{(1)} \underline{U}_{I}^{m}{}_{(1)} + \underline{\underline{M}}_{I\Gamma}^{(1)} \underline{U}_{\Gamma}^{m} - \underline{\underline{A}}_{I\Gamma}^{(1)} \underline{U}_{\Gamma}^{m+1, i} \right)$$

- For the second (right) subdomain, instead, we have:

$$\begin{cases} \dfrac{\partial u^{(2)^{i+1}}}{\partial t} - k \dfrac{\partial^2 u^{(2)^{i+1}}}{\partial x^2} = f^{(2)} \\ k \dfrac{\partial u^{(2)^{i+1}}}{\partial x} = -k \dfrac{\partial u^{(1)^{i+1}}}{\partial x} \quad \text{on} \quad \Gamma_{21} \; (x = 0.4^+) \\ u^{(2)} = 0 \quad \text{on } x = 1 \end{cases} \longrightarrow \begin{bmatrix} \text{NEUMANN} \\ \text{PROBLEM} \end{bmatrix}$$

which brings to the following algebraic version:

$$\begin{bmatrix} \underline{\underline{A}}_{II}^{(2)} & \underline{\underline{A}}_{I\Gamma}^{(2)} \\ \underline{\underline{A}}_{\Gamma I}^{(2)} & \underline{\underline{A}}_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} \underline{U}_I^{(2)^{i+1, m+1}} \\ \underline{U}_\Gamma^{i+1, m+1} \end{bmatrix} = \begin{bmatrix} \underline{f}_I^{(2)} \\ \underline{f}_\Gamma^{(2)} - \phi^{i+1, m+1} \end{bmatrix} + \begin{bmatrix} \underline{\underline{M}}_{II}^{(2)} & \underline{\underline{M}}_{I\Gamma}^{(2)} \\ \underline{\underline{M}}_{\Gamma I}^{(2)} & \underline{\underline{M}}_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} \underline{U}_I^{(2)^m} \\ \underline{U}_\Gamma^{m} \end{bmatrix}$$

With this scheme now each subdomain problem must be solved sequentially, firstly the Dirichlet problem, then the Neumann, everything concentrating on the same time step.

Problem 7

Writing in a compact way the original weak form of the problem in the left subdomain 1:

$$(\partial_t u_1, \delta u) + (\nabla u_1, \nabla \delta u) - (\delta u, \underline{n} \cdot \underline{\nabla} u_1)_{\Gamma_{12}} = (f, \delta u)$$

Nitsche's method adds a penalty term to make the system symmetric and consistent. This will brings to:

$$(\partial_t u_1, \delta u) + (\nabla u_1, \nabla \delta u) - (\delta u, \underline{n} \cdot \underline{\nabla} u_1)_{\Gamma_{12}} + \alpha (\delta u, u_1)_{\Gamma_{12}} - (u_1, \underline{n} \cdot \underline{\nabla} \delta u)_{\Gamma_{12}} =$$

$$= (f, \delta u) + \alpha (\delta u, u_2)_{\Gamma_{21}} - (u_2, \underline{n} \cdot \underline{\nabla} \delta u)_{\Gamma_{21}}$$

$\downarrow$

this creates the following additional matrices:

- $\alpha (\delta u, u_1)_{\Gamma_{12}} \rightarrow \alpha \sum_i^{N^\Gamma} \sum_j^{N^\Gamma} N_i N_j u_{1j}^{n+1} = \alpha \underline{\underline{M}}_B^{(1)} \underline{U}_\Gamma^{(1)^{n+1}}$
- $(u_1, \underline{n} \cdot \underline{\nabla} \delta u)_{\Gamma_{12}} \rightarrow \sum_i^{N^\Gamma} \sum_j^{N^\Gamma} N_i' N_j u_{1j}^{n+1} = \underline{\underline{N}}^{(1)T} \underline{U}_\Gamma^{(1)^{n+1}}$

Similarly the RHS terms will be: $\alpha \underline{\underline{M}}_B \underline{U}_\Gamma^{(2)^n}$, $\underline{\underline{N}}^{(1)T} \underline{U}_\Gamma^{(2)^n}$

And: $(\delta u, \underline{n} \cdot \underline{\nabla} u_1)_{\Gamma_{12}} \rightarrow \sum_i^{N^\Gamma} \sum_j^{N^\Gamma} N_i N_j' u_{1j}^{n+1} = \underline{\underline{N}}^{(1)} \underline{U}_\Gamma^{(1)^{n+1}}$

This will bring to the following problem: $(\Delta t = 1)$

$$(\underline{\underline{M}}^{(1)} + \underline{\underline{K}}^{(1)} + \underline{\underline{\hat{P}}}^{(1)}) \underline{U}^{(1)^{n+1}} = \underline{f}^{(1)} + \underline{\underline{M}}^{(1)} \underline{U}^{(1)^n} + \underline{\underline{\tilde{P}}}^{(1)} \underline{U}^{(2)^{n+1}}$$

where $\underline{\underline{\hat{P}}}^{(1)} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & -\underline{\underline{N}} + \alpha \underline{\underline{M}}_B - \underline{\underline{N}}^T \end{bmatrix}$ and

and $\underline{\underline{\tilde{P}}}^{(1)} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \alpha \underline{\underline{M}}_B - \underline{\underline{N}}^T \end{bmatrix}$

in a detailed way the system is:

$$\begin{bmatrix} \underline{\underline{A}}^{(1)}_{II} & \underline{\underline{A}}^{(1)}_{I\Gamma} \\ \underline{\underline{A}}^{(1)}_{\Gamma I} & \underline{\underline{A}}^{(1)}_{\Gamma\Gamma} - \underline{\underline{N}} + \alpha \underline{\underline{M}}_B - \underline{\underline{N}}^T \end{bmatrix} \begin{bmatrix} \underline{U}_I^{(1)\, n+1} \\ \underline{U}_\Gamma^{n+1} \end{bmatrix} = \begin{bmatrix} \underline{f}_I^{(1)} \\ \underline{f}_\Gamma^{(1)} \end{bmatrix} + \begin{bmatrix} \underline{\underline{M}}^{(1)}_{II} & \underline{\underline{M}}^{(1)}_{I\Gamma} \\ \underline{\underline{M}}^{(1)}_{\Gamma I} & \underline{\underline{M}}^{(1)}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \underline{U}_I^{(1)\, n} \\ \underline{U}_\Gamma^{n} \end{bmatrix} +$$

$$\begin{bmatrix} \underline{\underline{O}} & \underline{\underline{O}} \\ \underline{\underline{O}} & \alpha \underline{\underline{M}}_B - \underline{\underline{N}}^T \end{bmatrix} \begin{bmatrix} \underline{U}_I^{(2)\, n+1} \\ \underline{U}_\Gamma^{n+1} \end{bmatrix}$$

coming from the other sbd

Exercise 5: OPERATOR SPLITTING TECHNIQUES

Considering the 1-dimensional, transient, convection-diffusion equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - k \dfrac{\partial^2 u}{\partial x^2} + a_x \dfrac{\partial u}{\partial x} = f & \text{in } [0,1] \\ u = 0 & \text{in } x = 0, \ \forall t \\ u = 0 & \text{in } x = 1, \ \forall t \\ u = 0 & \text{in } t = 0, \ \forall x \in [0,1] \end{cases}$$

with $k = 1$, $a_x = 1$, $f = 1$

Problem 1

As in the previous exercise, let us do the temporal discretization first: showing first the θ-method and then imposing $\theta = 1$ and a first order expansion:

$$\frac{u^{n+1} - u^n}{\Delta t} = \theta \left(\frac{\partial u}{\partial t}\right)^{n+1} + (1-\theta)\left(\frac{\partial u}{\partial t}\right)^n$$

let us call $\Delta u = u^{n+1} - u^n$, $\Delta u_t = \left(\frac{\partial u}{\partial t}\right)^{n+1} - \left(\frac{\partial u}{\partial t}\right)^n$

then we have:

$$\frac{\Delta u}{\Delta t} - \theta \Delta u_t = u_t^n \quad , \text{ where } u_t^n = \left(\frac{\partial u}{\partial t}\right)^n$$

Since $\dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2} - a_x \dfrac{\partial u}{\partial x} + f$, and calling: $\Delta u_{xx} = \left(\frac{\partial^2 u}{\partial x^2}\right)^{n+1} - \left(\frac{\partial^2 u}{\partial x^2}\right)^n$

and $\Delta u_x = \left(\frac{\partial u}{\partial x}\right)^{n+1} - \left(\frac{\partial u}{\partial x}\right)^n$, $\Delta f = f^{n+1} - f^n$

We obtain :

$$\frac{\Delta u}{\Delta t} - \theta k \, \Delta u_{xx} + \theta a_x \, \Delta u_x - \theta \Delta f = k \, u_{xx}^n - a_x u_x^n + f^n$$

using $\theta = 1$, $k = 1$, $a_x = 1$, $f = 1$,

$$\boxed{\frac{\Delta u}{\Delta t} - \Delta u_{xx} + \Delta u_x = \underbrace{u_{xx}^n - u_x^n + 1}_{\text{Constant}}}$$ equation "1"

which is equal to :

$$\frac{\Delta u}{\Delta t} - u_{xx}^{n+1} + \cancel{u_{xx}^n} + u_x^{n+1} - \cancel{u_x^n} = \cancel{u_{xx}^n} - \cancel{u_x^n} + 1$$

So : $$\boxed{\frac{\Delta u}{\Delta t} - \left(\frac{\partial^2 u}{\partial x^2}\right)^{n+1} + \left(\frac{\partial u}{\partial x}\right)^{n+1} = 1}$$ equation "2"

- Using the finite element discretization, starting from using a test function δu such that $\delta u = 0$ on $x = 0$ and $x = 1$:

let us use eq 1 for simplicity

$$\int_0^1 \delta u \frac{\Delta u}{\Delta t} dx - \int_0^1 \delta u \frac{d^2 \Delta u}{dx^2} dx + \int_0^1 \delta u \frac{d \Delta u}{dx} dx = \int_0^1 \delta u \, dx + \int_0^1 \delta u \left(\frac{d^2 u}{dx^2}\right)^n dx -$$

$$- \int_0^1 \delta u \left(\frac{du}{dx}\right)^n dx$$

$$= \int_0^1 \delta u \frac{\Delta u}{\Delta t} dx - \underbrace{\cancel{\delta u \frac{d \Delta u}{dx}}}_{\emptyset \, (b.c.)} \Big|_{x=0/1} + \int_0^1 \frac{d \delta u}{dx} \frac{d \Delta u}{dx} dx +$$

$$+ \int_0^1 \delta u \frac{d \Delta u}{dx} dx = \int_0^1 \delta u \, dx + \underbrace{\cancel{\delta u \left(\frac{du}{dx}\right)^n}}_{\emptyset \, (b.c.)} \Big|_{x=0/1} - \int_0^1 \frac{d \delta u}{dx} \left(\frac{du}{dx}\right)^n dx -$$

$$- \int_0^1 \delta u \left(\frac{du}{dx}\right)^n dx$$

Imaging size $h = \frac{1}{3}$, such that $\underline{U} \cong \underline{U}_{1/3}, \equiv \sum_{j=1}^{4} u_j N_j(x)$

$\delta u \cong \delta u_{1/3} \equiv N_i(x)$

4: number of nodes (total)

we will have in a compact way:

$$\left(\frac{\Delta u}{\Delta t}, \delta u\right) + \left(\frac{d\Delta u}{dx}, \frac{d\delta u}{dx}\right) + \left(\frac{d\Delta u}{dx}, \delta u\right) = (1, \delta u) - \left(\left(\frac{du}{dx}\right)^n, \frac{d\delta u}{dx}\right)$$

$$-\left(\frac{du^n}{dx}, \delta u\right) \quad \forall\, u, \delta u \in H_0^1([0,1])$$

$\Downarrow$

first derivative

$$\sum_{j=1}^{4}\left(N_i, N_j \frac{\Delta u_j}{\Delta t}\right)_\Omega + \sum_{j=1}^{4}\left(N_i', N_j' \Delta u_j\right)_\Omega + \sum_{j=1}^{4}\left(N_i, N_j' \Delta u_j\right)_\Omega = \sum_{j=1}^{4}(N_i, 1)_\Omega$$

$$- \sum_{j=1}^{4}\left(N_i', N_j' u_j^n\right)_\Omega - \sum_{j=1}^{4}\left(N_i, N_j' u_j^n\right)_\Omega$$

Which can be sintetized as follows:

$$\underline{\underline{M}} \frac{\Delta \underline{U}}{\Delta t} + \underline{\underline{K}}\, \Delta\underline{U} + \underline{\underline{C}}\, \Delta \underline{U} = \underline{f}_1 - \underline{\underline{K}}\, \underline{U}^n - \underline{\underline{C}}\, \underline{U}^n$$

Knowing that: $\Delta \underline{U} = \underline{U}^{n+1} - \underline{U}^n$

we have:

$$\left(\frac{\underline{\underline{M}}}{\Delta t} + \underline{\underline{K}} + \underline{\underline{C}}\right) \underline{U}^{n+1} = \underline{F}_1 + \frac{\underline{\underline{M}}}{\Delta t} \underline{U}^n$$

where $\underline{\underline{M}}$ is mass matrix, $\underline{\underline{K}}$ is stiffness matrix, $\underline{\underline{C}}$ is the convective matrix

In our case, let us use three-element division, focus on each element, and then assemble

$L_e = 1/3$, and the isoparametric shape functions are:

$$\forall \text{ element} \begin{cases} N_1(\xi) = \frac{1}{2}(1-\xi) \\ N_2(\xi) = \frac{1}{2}(1+\xi) \end{cases}, \quad \xi \in [-1, 1]$$

$$\frac{dN_1}{dx} = N_1'(x) = \frac{dN_1}{d\xi}\cdot\frac{d\xi}{dx} = -\frac{1}{2}\cdot\frac{2}{L_e} = -\frac{1}{L_e} = -\frac{1}{1/3} = -3$$

$$\frac{dN_2}{dx} = N_2'(x) = \frac{dN_2}{d\xi}\cdot\frac{d\xi}{dx} = \frac{1}{2}\cdot\frac{2}{L_e} = \frac{1}{L_e} = \frac{1}{1/3} = 3$$

$$|\det \underline{J}| = \left|\frac{dx}{d\xi}\right| = \frac{L^e}{2}$$

$\forall$ element:

$$\underline{\underline{M}}^e = \frac{L^e}{2}\int_{-1}^{1}\begin{bmatrix}\frac{1}{2}(1-\xi)\\ \frac{1}{2}(1+\xi)\end{bmatrix}\begin{bmatrix}\frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi)\end{bmatrix}d\xi = \begin{bmatrix}1/9 & 1/18\\ 1/18 & 1/9\end{bmatrix}$$

$$\underline{\underline{K}}^e = \frac{L^e}{2}\int_{-1}^{1}\begin{bmatrix}-\frac{1}{L^e}\\ \frac{1}{L^e}\end{bmatrix}\begin{bmatrix}-\frac{1}{L^e} & \frac{1}{L^e}\end{bmatrix}d\xi = \begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix}$$

$$\underline{\underline{C}}^e = \frac{L^e}{2}\int_{-1}^{1}\begin{bmatrix}\frac{1}{2}(1-\xi)\\ \frac{1}{2}(1+\xi)\end{bmatrix}\begin{bmatrix}-\frac{1}{L^e} & \frac{1}{L^e}\end{bmatrix}d\xi = \begin{bmatrix}-1/2 & 1/2\\ -1/2 & 1/2\end{bmatrix}$$

After computing the assembly process, we shall have:

$$\left(\frac{1}{\Delta t}\underbrace{\begin{bmatrix}1/9 & 1/18 & 0 & 0\\ 1/18 & 2/9 & 1/18 & 0\\ 0 & 1/18 & 2/9 & 1/18\\ 0 & 0 & 1/18 & 1/9\end{bmatrix}}_{\underline{\underline{M}}} + \underbrace{\begin{bmatrix}3 & -3 & 0 & 0\\ -3 & 6 & -3 & 0\\ 0 & -3 & 6 & -3\\ 0 & 0 & -3 & 3\end{bmatrix}}_{\underline{\underline{K}}} + \underbrace{\begin{bmatrix}-\frac{1}{2} & \frac{1}{2} & 0 & 0\\ -\frac{1}{2} & 0 & \frac{1}{2} & 0\\ 0 & -\frac{1}{2} & 0 & \frac{1}{2}\\ 0 & 0 & -\frac{1}{2} & \frac{1}{2}\end{bmatrix}}_{\underline{\underline{C}}}\right)\underbrace{\begin{bmatrix}u_0\\ u_1\\ u_2\\ u_3\end{bmatrix}^{n+1}}_{\underline{U}^{n+1}} =$$

$$\underbrace{\begin{bmatrix}1/6\\ 2/6\\ 2/6\\ 1/6\end{bmatrix}}_{\underline{F_1}} + \frac{1}{\Delta t}\underbrace{\begin{bmatrix}1/9 & 1/18 & 0 & 0\\ 1/18 & 2/9 & 1/18 & 0\\ 0 & 1/18 & 2/9 & 1/18\\ 0 & 0 & 1/18 & 1/9\end{bmatrix}}_{\underline{M}}\underbrace{\begin{bmatrix}u_0\\ u_1\\ u_2\\ u_3\end{bmatrix}^{n}}_{\underline{U}^n}$$

Imposing the external boundary conditions (Dirichlet), $u_0 = 0$ and $u_3 = 0$ $\forall n \in \mathbb{N}$

We eliminate 1st and 4th rows and columns:

$$\begin{bmatrix} \frac{2}{9}\cdot\frac{1}{\Delta t}+6 & \frac{1}{18}\cdot\frac{1}{\Delta t}-\frac{5}{2} \\ \frac{1}{18}\cdot\frac{1}{\Delta t}-\frac{7}{2} & \frac{2}{9}\cdot\frac{1}{\Delta t}+6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^{n+1} = \begin{bmatrix} 2/6 \\ 2/6 \end{bmatrix} + \begin{bmatrix} 2/9 & 1/18 \\ 1/18 & 2/9 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^{n}$$

Imposing that $\underline{U}_{t=0} = \underline{0}$, we have

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^{1} = \begin{bmatrix} \frac{2}{9}\cdot\frac{1}{\Delta t}+6 & \frac{1}{18}\cdot\frac{1}{\Delta t}-\frac{5}{2} \\ \frac{1}{18}\cdot\frac{1}{\Delta t}-\frac{7}{2} & \frac{2}{9}\cdot\frac{1}{\Delta t}+6 \end{bmatrix}^{-1} \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \frac{6\Delta t(51\Delta t+1)}{2943\Delta t^2+324\Delta t+5} \\ \frac{6\Delta t(57\Delta t+1)}{2943\Delta t^2+324\Delta t+5} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^{1}$$

Problem 2

The first order convection-diffusion operator splitting is the following:

we know that: $u^n = u^n_{conv} = u^n_{diff}$

- this means, firstly solve for u^{n+1}_{conv}

$$\partial_t u^{n+1}_{conv} + \mathcal{L}_{conv} u^{n+1}_{conv} = 0$$

where $\mathcal{L}_{conv} u_{conv} = a_x \frac{du_{conv}}{dx}$ (in this case $a_x = 1$)

in algebraic form:

Solve $\underline{U}^{n+1}_{conv}$: $\left(\frac{\underline{\underline{M}}}{\Delta t} + \underline{\underline{C}}\right)\underline{U}^{n+1}_{conv} = \frac{\underline{\underline{M}}}{\Delta t}\underline{U}^{n}_{conv}$

- Secondly, imposing that $u^n_{diffusive} = u^{n+1}_{conv}$

Solve for u^{n+1}_{diff}

$$\partial_t u^{n+1}_{diff} + \mathcal{L}_{diff} u^{n+1}_{diff} = f$$ → in this case $f = 1$

where $\mathcal{L}_{diff} u_{diff} = -k \frac{d^2 u_{diff}}{dx^2}$ (in this case $k = 1$)

in algebraic form:

$$\left(\frac{\underline{\underline{M}}}{\Delta t} + \underline{\underline{K}}\right)\underline{U}^{n+1}_{diff} = \underline{f}_1 + \frac{\underline{\underline{M}}}{\Delta t}\underline{U}^{n}_{diff}$$

Coming from the convective problem

- finally update the solution as follows:

$$\underline{U}^{n+1} = \underline{U}^{n+1}_{diff}$$

Imposed already the Dirichlet b.c.:

(initial conditions)

1)
$$\begin{bmatrix} \frac{2}{9}\cdot\frac{1}{\Delta t} & \frac{1}{18}\cdot\frac{1}{\Delta t}+\frac{1}{2} \\ \frac{1}{18}\cdot\frac{1}{\Delta t}-\frac{1}{2} & \frac{2}{9}\cdot\frac{1}{\Delta t} \end{bmatrix}\begin{bmatrix} u_{1conv} \\ u_{2conv} \end{bmatrix}^{n+1} = \frac{1}{\Delta t}\cancel{\begin{bmatrix} 2/9 & 1/18 \\ 1/18 & 2/9 \end{bmatrix}}\cancel{\begin{bmatrix} u^{n}_{1conv} \\ u^{n}_{2conv} \end{bmatrix}}_{\underline{0}}$$

$n = 0 \rightarrow$

$$\rightarrow \begin{bmatrix} u_{1conv} \\ u_{2conv} \end{bmatrix}^{1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2)
$$\begin{bmatrix} \frac{2}{9}\cdot\frac{1}{\Delta t}+6 & \frac{1}{18}\cdot\frac{1}{\Delta t}-3 \\ \frac{1}{18}\cdot\frac{1}{\Delta t}-3 & \frac{2}{9}\cdot\frac{1}{\Delta t}+6 \end{bmatrix}\begin{bmatrix} u_{1diff} \\ u_{2diff} \end{bmatrix}^{n+1} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} + \underbrace{\frac{1}{\Delta t}\cancel{\begin{bmatrix} 2/9 & 1/18 \\ 1/18 & 2/9 \end{bmatrix}}\overbrace{\begin{bmatrix} u^{(1)}_{1conv} \\ u^{(1)}_{2conv} \end{bmatrix}}^{\underline{0}}}_{\underline{\emptyset}}$$

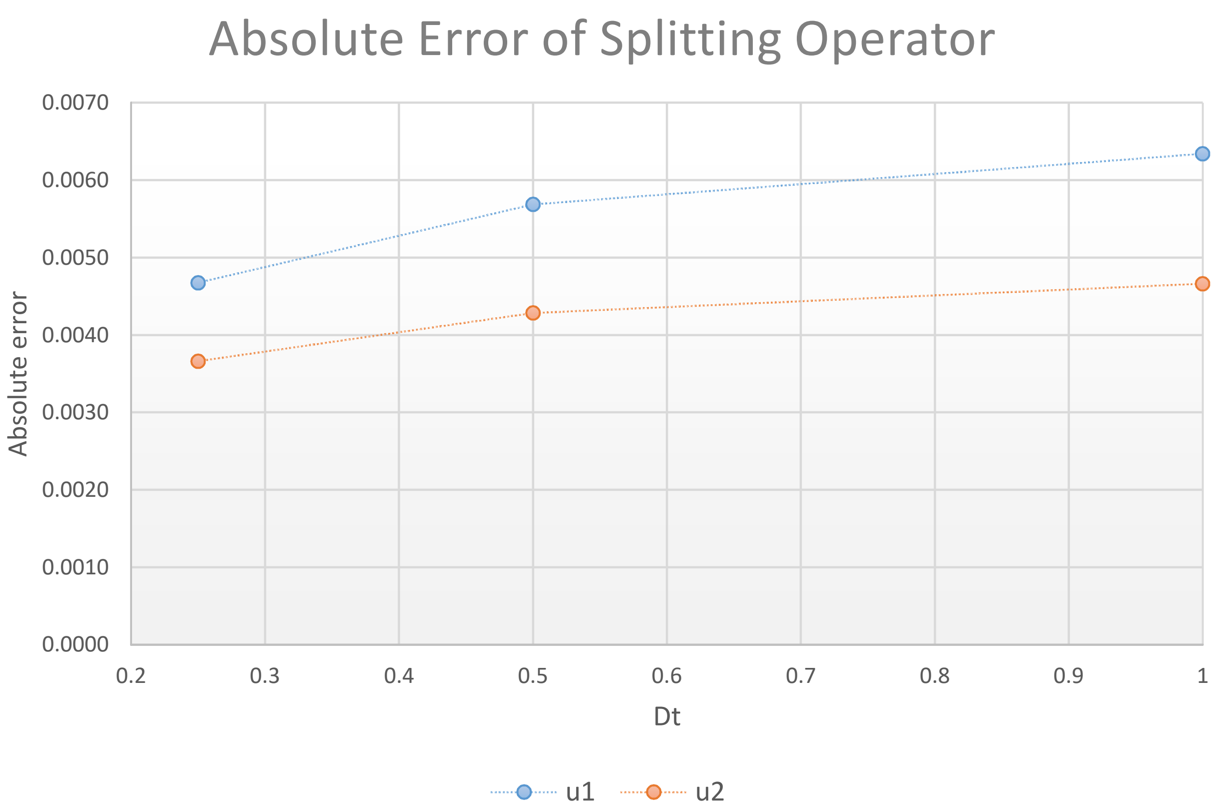
So doing:
$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^{n+1} = \begin{bmatrix} u_{1diff} \\ u_{2diff} \end{bmatrix}^{n+1} = \begin{bmatrix} \frac{2}{9}\cdot\frac{1}{\Delta t}+6 & \frac{1}{18}\cdot\frac{1}{\Delta t}-3 \\ \frac{1}{18}\cdot\frac{1}{\Delta t}-3 & \frac{2}{9}\cdot\frac{1}{\Delta t}+6 \end{bmatrix}^{-1}\begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{6\Delta t}{54\Delta t+5} \\ \frac{6\Delta t}{54\Delta t+5} \end{bmatrix}$$

Exercise 5.3 - Results from the calculation

	Solution					
	Monolitic			*Splitting Operator*		
Δt	**0.25**	**0.5**	**1**	**0.25**	**0.5**	**1**
u_1	0.0764	0.0881	0.0954	0.0811	0.0938	0.1017
u_2	0.0847	0.0980	0.1064	0.0811	0.0938	0.1017

	Errors					
	Absolute			*Differential (split. - mon.)*		
Δt	**0.25**	**0.5**	**1**	**0.25**	**0.5**	**1**
u_1	0.0047	0.0057	0.0063	0.0047	0.0057	0.0063
u_2	0.0037	0.0043	0.0047	-0.0037	-0.0043	-0.0047



Problem 3

As schown in the ~~[illegible]~~ previous table and in the previous results for the case of use of operator splitting techniques, the solution is the same in both nodes 1 and 2. This is due to the error given by the negligence of the convective term and the source $\underline{f}$ is applied directly in the second step for the "only diffusive" problem. Therefore, since the convection is not playing, the solution remains symmetric.

As shown in the table, as for the error (with sign), it's possible to see that the lack of trasport shown in the operator splitting technique, presents an excessive value in node 1, while it is smaller than the monolitic in the node 2. This errors, both for absolute and with sign, decrease with the size of the time interval Δt

---

EXERCISE 6: FRACTIONAL STEP METHODS

Problem 1

Given the following Yosida scheme for the incompressible Navier-Stokes equations:

- $$\underline{\underline{M}}\frac{1}{\delta t}(\underline{\hat{U}}^{n+1}-\underline{U}^{n})+\underline{\underline{K}}\,\underline{\hat{U}}^{n+1}=\underline{f}-\underline{\underline{G}}\,\underline{\hat{P}}^{n+1}$$

- $$\underline{\underline{D}}\,\underline{\underline{M}}^{-1}\underline{\underline{G}}\,\underline{P}^{n+1}=\frac{1}{\delta t}\underline{\underline{D}}\,\underline{\hat{U}}^{n+1}-\underline{\underline{D}}\,\underline{\underline{M}}^{-1}\underline{\underline{G}}\,\underline{\hat{P}}^{n+1}$$

- $$\underline{\underline{M}}\frac{1}{\delta t}(\underline{U}^{n+1}-\underline{\hat{U}}^{n+1})+\alpha\underline{\underline{K}}(\underline{U}^{n+1}-\underline{\hat{U}}^{n+1})+\underline{\underline{G}}(\underline{P}^{n+1}-\underline{\hat{P}}^{n+1})=\underline{0}$$

let us "open" the first and the third equation, we shall have:

$$\text{1}\quad \frac{\underline{\underline{M}}}{\delta t}\underline{\hat{U}}^{n+1} - \frac{\underline{\underline{M}}}{\delta t}\underline{U}^{n} + \underline{\underline{K}}\,\underline{\hat{U}}^{n+1} + \underline{\underline{G}}\,\underline{\hat{P}}^{n+1} = \underline{f}$$

$$\text{3}\quad \frac{\underline{\underline{M}}}{\delta t}\underline{U}^{n+1} - \frac{\underline{\underline{M}}}{\delta t}\underline{\hat{U}}^{n+1} + \alpha\underline{\underline{K}}\,\underline{U}^{n+1} - \alpha\underline{\underline{K}}\,\underline{\hat{U}}^{n+1} + \underline{\underline{G}}\,\underline{P}^{n+1} - \underline{\underline{G}}\,\underline{\hat{P}}^{n+1} = \underline{0}$$

Adding them, we shall have:

$$\frac{\underline{\underline{M}}}{\delta t}\underline{U}^{n+1} - \cancel{\frac{\underline{\underline{M}}}{\delta t}\underline{\hat{U}}^{n+1}} + \cancel{\frac{\underline{\underline{M}}}{\delta t}\underline{\hat{U}}^{n+1}} - \frac{\underline{\underline{M}}}{\delta t}\underline{U}^{n} + \alpha\underline{\underline{K}}\,\underline{U}^{n+1} - \alpha\underline{\underline{K}}\,\underline{\hat{U}}^{n+1} + \underline{\underline{K}}\,\underline{\hat{U}}^{n+1} + \underline{\underline{G}}\,\underline{P}^{n+1} - \cancel{\underline{\underline{G}}\,\underline{\hat{P}}^{n+1}} + \cancel{\underline{\underline{G}}\,\underline{\hat{P}}^{n+1}} = \underline{f}$$

$$= \frac{\underline{\underline{M}}}{\delta t}\left(\underline{U}^{n+1} - \underline{U}^{n}\right) + \alpha\underline{\underline{K}}\,\underline{U}^{n+1} + (1-\alpha)\underline{\underline{K}}\,\underline{\hat{U}}^{n+1} + \underline{\underline{G}}\,\underline{P}^{n+1} = \underline{f}$$

it's possible to see that for $\alpha = 1$

we recover the original Momentum equation

Problem 2

The Yosida method has been introduced to deal with unsteady incompressible Navier-Stokes equation. It Does a splitting of the original problem into smaller problems, separating the velocity and the pressure fields. This method is a "quasi-compressibility" scheme since the splitting of the problem introduces an error affecting the continuity equation. In this scheme a small perturbation is added to the continuity equation in order to stabilize the solution, but it must be done wisely so that linear elements can be used for both fields approximations

There are three possible suitable corrections:

1) $\underline{\nabla} \cdot \underline{u} + \varepsilon \frac{\partial P}{\partial t} = 0$, $P\big|_{t=0} = P_0$ as ARTIFICIAL COMPRESSIBILITY METHOD

2) $\underline{\nabla} \cdot \underline{u} + \varepsilon P = 0$ banal PENALTY METHOD

3) $\underline{\nabla} \cdot \underline{u} + \varepsilon \Delta P = 0$ PETROV - GALERKIN METHOD (ELLIPTIC)
with $\underline{\nabla} P \cdot \underline{n}\big|_{\Gamma} = 0$

ε can't be too small otherwise it doesn't stabilizes enough the numerical method, but can't be large either, otherwise the perturbations given are not enough minimized in the continuity equation.

Exercise 7: ALE FORMULATIONS

Problem 1

Given the spatial description of a property:

$$\gamma(x, y, z, t) = \begin{bmatrix} 2x \\ y e^t \\ z \end{bmatrix}$$

The equations of movement:

$$\begin{cases} x = X e^t \\ y = Y + e^t - 1 \\ z = Z \end{cases}$$

The equations of the movement of the mesh:

$$\begin{cases} x_m = \chi + \alpha t \\ y_m = \chi - \beta t \\ z_m = \chi \end{cases}$$

Question (a)

The mapping between the mesh nodes $\underline{\chi}$ and the spatial coordinates $\underline{x}$ is $\underline{x} = \phi(\underline{\chi}, t) = \underline{x}(\underline{\chi}, t)$

The ALE coordinates will be then:

$$\gamma_{ALE} = \gamma(\underline{x}, t) = \gamma(\phi(\underline{X}, t)) = \begin{bmatrix} 2(X+\alpha t) \\ (Y-\beta t)e^t \\ Z \end{bmatrix}$$

Question (b)

the velocity of the particles will be:

$$\underline{v}(\underline{X}, t) = \frac{\partial \underline{x}}{\partial t} = \begin{bmatrix} X e^t \\ e^t \\ 0 \end{bmatrix}$$

the velocity of the mesh will be:

$$\underline{v}_m(\underline{X}, t) = \frac{\partial \underline{x}(\underline{X}, t)}{\partial t} = \begin{bmatrix} \alpha \\ -\beta \\ 0 \end{bmatrix}$$

Question (c)

The ALE description of the material derivative in time will be:

$$\frac{d\gamma_{ALE}}{dt} = \frac{d}{dt}\gamma_{ALE}(\underline{X}(\underline{X}, t), t) = \frac{\partial \gamma_{ALE}(\underline{X}, t)}{\partial t} + (\underline{v} - \underline{v}_m)\cdot \underline{\nabla}\gamma(\underline{x}, t)$$

where: $$\frac{\partial \gamma_{ALE}}{\partial t} = \frac{\partial}{\partial t}\begin{bmatrix} 2(X+\alpha t) \\ (Y-\beta t)e^t \\ Z \end{bmatrix} = \begin{bmatrix} 2\alpha \\ (Y-\beta-\beta t)e^t \\ \emptyset \end{bmatrix}$$

$$\underline{v} - \underline{v}_m = \begin{bmatrix} X e^t - \alpha \\ e^t + \beta \\ 0 \end{bmatrix} = \begin{bmatrix} X + \alpha(t-1) \\ e^t + \beta \\ 0 \end{bmatrix}$$

$$\underline{\nabla}\underline{\gamma} = \left[\frac{\partial \gamma_i}{\partial x_j}\right] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally we obtain: $\frac{d}{dt}\gamma_{ALE}(\underline{X}(\underline{X}, t), t) =$ (see in the next page)

$$= \begin{bmatrix} 2\alpha \\ (Y - \beta t - \beta)e^t \\ 0 \end{bmatrix} + \begin{bmatrix} Xe^t - \alpha \\ e^t + \beta \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cancel{2\alpha} + 2Xe^t - \cancel{2\alpha} \\ (Y - \beta t - \cancel{\beta})e^t + e^{2t} + \cancel{\beta e^t} \\ 0 \end{bmatrix} = \begin{bmatrix} 2Xe^t \\ (Y - \beta t + e^t)e^t \\ 0 \end{bmatrix}$$

Problem 2

The ALE for of incompressible Navier-Stokes equation must be obtained from the classical form. This means to modify the convective term using the "convective velocity" $\underline{c}$: $\underline{c} = \underline{u} - \underline{u}_{mesh}$ instead of the normal eulerian from $\underline{u}$

- MASS EQUATION: $\underbrace{\partial_t \rho(\underline{X}, t)}_{\text{calculated at the MESH POSITION}} + \underbrace{\underline{c} \cdot \underline{\nabla} \rho = -\rho \underline{\nabla} \cdot \underline{u}}_{\text{calculated at the SPATIAL COORDINATES}}$

- INCOMPRESSIBILITY: $\underline{\nabla} \cdot \underline{u} = 0$

- MOMENTUM EQUATION: $\rho\left(\underbrace{\partial_t \underline{v}(\underline{X}, t)}_{\text{calculated at the mesh position}} + \underbrace{\underline{c}(\underline{\nabla}\underline{u})\right) = \underline{\nabla} \cdot \underline{\underline{\sigma}}}_{\text{calculated at the spatial COORDINATES}} + \rho \underline{b}$

Since there is incompressibility, $\rho =$ constant; plus, if we consider a Newtonian fluid we would get that $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \nu \underline{\nabla}^s \underline{u}$

So at last, the momentum equation would be like:

$$\boxed{\partial_t \underline{v}(\underline{X}, t) + \underline{c}(\underline{\nabla}\underline{u}) = -\frac{1}{\rho}\underline{\nabla} p + \nu \Delta \underline{u} + \underline{b}}$$

Eulerian (spatial)
which contains evaluation of the quantities everywhere except for the first term which represents a ~~term~~ quantity evaluated in the mixed reference (ALE).

The time discretization can be done with any ~~a~~ method with finite differences for the first, mesh referenced, term. The other, spatial terms, are computed in spatial coordinates. This means that the temporal derivative, once computed in finite differences way, calculates the difference, in Δt time interval, from previous step (n) to following one (n+1) concentrating at the moving nodes of the mesh.

Problem 3 – Bibliographical research

The update of the movement of the mesh can be done in two different ways: 1] mesh regularization; 2] mesh adaptation:

1] Mesh regularization:

this method consists in keeping the mesh as regular as possible in order to avoid mesh entanglement. This requires an updated information about the mesh conditions at each time step (like position of the nodes) from the dispacements just calculated or from the current mesh velocity. There are many ways to compute this:

I) If the motion (velocity) of the material surfaces (usually the boundary) is known ~~at the~~ a priori, the mesh motion is also prescribed a priori. This can be done by giving a well defined mesh velocity. This makes a formulation of Lagrangian type for the moving surfaces, and an Eulerian formulation done for away from the boundaries.

This is well explained in two main works: HUERTA AND LIU (1988a) and RODRIGUEZ-FERRAN (2002) et. al.

II) Mainly, in all other cases, where the motion is unknown and not prescribed, the position of a layer or boundary (or at least along the normal direction) must be tracked during the computation. This imposes of course a Lagrangian formulation along this boundary. Different possibility is for fluid-structure interaction, where the structure is treated with Lagrangian formulation while the fluid has fixed mesh or at the most updated through simple interpolation. About the first general case, there are in bibliography some numerical computation done i.e. by NOH in 1964. 20 years later by Liu and Chang (1984) a more deep study has been done using fluid-Structure interaction.

Extra studies and Techniques are the following:

III) Using a "transfinite mapping" method: developed in 1991 and showed by Ponthot and Hogge, it involves a cheap procedure by which creates a mesh from boundaries once the have been discretized (depending on the dimension)

IV) Mesh Smoothing: it is an algorithm by which the system minimizes both squeeze and distorsion of each element (ex: Donea in 1982)

V) Laplacian Smoothing: this is a smart method since it involves the laplacian/Poisson problem in order to rezone properly the nodes. (Liu in 1988)

2] The second possibility of developing a good ALE formulation is the MESH-ADAPTATION method: here the aim is to optimize the mesh to achieve a finer accuracy, using a cheap computing cost.

This means that the number of the elements and the connectivity are constant, but moves (adapt) the nodes of the mesh where there is a strong gradient (like large deformation for non-linear mechanics). The error indicates the need of modify or not, and it is set such that there is an equilibrate distribution in the domain (this is done by using elliptic or parabolic differential equation). Again HUERTA (1997) and RODRÍGUEZ-FERRAN (1998) HARM, are the main works found on this subject.

bibliography (in order of quote)

1) HUERTA & LIU (1988a) - Viscous flow structure interaction - Journal of pressure vessel technology

2) RODRIGUEZ-FERRAN (2002) - Arbitrary Lagrangian - eulerian (ale) formulation for hyperelasticity - Journal International for Numerical Methods in Engineering

3) NoH (1964) - A time-dependent two-space dimensional coupled Eulerian-Lagrangian code - Int. Methods in Computational Physics, Alder B, Fernbach S and Rotenberg M (eds), vol. 3 Academic Press: New York

4) LIU & CHANG (1984) - Efficient computational procedures for long-time duration fluid-structure interaction problems - J. Press. Vessel Technol. - Trans. ASME 1984

5) PONTHOT & HOGGE (1991) - The use of the Eulerian-Lagrangian FEM in metal forming applications including contact and adaptive mesh. - In Advances in Finite Deformation Problems in Material Processing, Chandra N and Reddy JN (eds). ASME Winter Annual Meeting, ASME: AMD-125, ASME (American Society of Mechanical Engineers), Atlanta 1991

6) DONEA, GIULIANI and HALLEUX - An arbitrary Lagrangian - Eu-

lerian finite element method for transient dynamic fluid-structure interactions. Computat. Methods Eng. 1982

7) LIU, CHANG, CHEN and BELYTSCHKO – Arbitrary Lagrangian-Eulerian Petrov-Galerkin finite elements for non linear continua – Computational Methods Applied Mechanics in Engineering – 1988

8) HUERTA, RODRIGUEZ-FERRAN, DIEZ AND SARRATE – Adaptive finite element strategies based on error assessment – International Journal for Numerical Methods in Engineering – 1999

9) HARM, RODRIGUEZ-FERRAN, HUERTA – Adaptive analysis of yield line patterns in plates with the arbitrary lagrangian-eulerian method – Computers & Structures, 1999

Exercise 8: FLUID-STRUCTURE INTERACTION

Problem 1

The added mass effect happens when the fluid density is similar to the structure density (for example in bio-mechanics, body tissues versus blood or water); the problem is in the fact that the partitioning schemes so far considered, do not work properly. The added mass operator is a description of the predicted acceleration at the interface related to the new forces at the interface coming from the fluid and applied to the structure. This mass is an additional contribution which is wrong: with Aitken's relaxation method, or steepest-descent method or Robin-Robin B.C. at the interface can decrease this problem

Problem 2

The original one-dimensional heat transfer problem to be solve is:

$$\begin{cases} \dfrac{\partial u}{\partial t} - k \dfrac{\partial^2 u}{\partial x^2} = f \\ u = 0 \quad \text{on } \partial\Omega \ \forall t \in \mathbb{R}^+ \end{cases}$$

After discretizing in space using FE and in time with BDF1 method we have:

$$\left(\frac{\underline{\underline{M}}}{\Delta t} + \underline{\underline{K}}\right) \underline{U}^{n+1} = \underline{f} + \frac{\underline{\underline{M}}}{\Delta t} \underline{U}^{n} \qquad (\text{with } k = f = 1)$$

Coming back to the continuous form we have to partition the domain in the following way:

- In subdomain 1,

  Solve Neumann Problem:

$$\begin{cases} \left(\frac{\underline{\underline{M}}^{(1)}}{\Delta t} + \underline{\underline{K}}^{(1)}\right) \underline{U}^{(1)^{n+1,i}} = \underline{f}^{(1)} + \frac{\underline{\underline{M}}^{(1)}}{\Delta t} \underline{U}^{(1)^{n}} & \text{in } \Omega_1 \\ \overset{1}{\cancel{k}} \dfrac{du^{n+1,i}_{\Gamma_{12}}}{dx} = - \overset{1}{\cancel{k}} \dfrac{du^{n+1,i-1}_{\Gamma_{21}}}{dx} & \text{on } \Gamma_{12} \\ u^{(1)\, n+1,i} = 0 & \text{on } \partial\Omega_1 \setminus \Gamma_{12} \end{cases}$$

- Calculate the Dirichlet value for the problem in subdomain 2, using Aitken's relaxation Scheme:

$$\omega = \frac{u^{n+1,i-2}_{\Gamma_{21}} - u^{n+1,i-1}_{\Gamma_{21}}}{u^{n+1,i-2}_{\Gamma_{21}} - u^{n+1,i-1}_{\Gamma_{21}} + u^{n+1,i}_{\Gamma_{12}} - u^{n+1,i-1}_{\Gamma_{12}}}$$

- In subdomain 2,

  Solve Dirichlet problem with relaxation factor ω:

$$\begin{cases} \left(\frac{\underline{\underline{M}}^{(2)}}{\Delta t} + \underline{\underline{K}}^{(2)}\right) \underline{U}^{(2)\, n+1,i} = \underline{f}^{(2)} + \frac{\underline{\underline{M}}^{(2)}}{\Delta t} \underline{U}^{(2)\, n} & \text{in } \Omega_2 \\ u^{n+1,i}_{\Gamma_{21}} = \omega u^{n+1,i}_{\Gamma_{12}} + (1-\omega) u^{n+1,i-1}_{\Gamma_{21}} & \text{on } \Gamma_{21} \\ u^{n+1,i}_{(2)} = 0 & \text{on } \partial\Omega_2 \setminus \Gamma_{21} \end{cases}$$

Using two iterations $i = 1, 2$ would give a convergent solution. But using Aitken's method, it's needed to use an $i > 2$, so the relaxation can be chosen constant with a value $\in (0, 1]$. The following pages an implemented Matlab code shows the results for this problem.

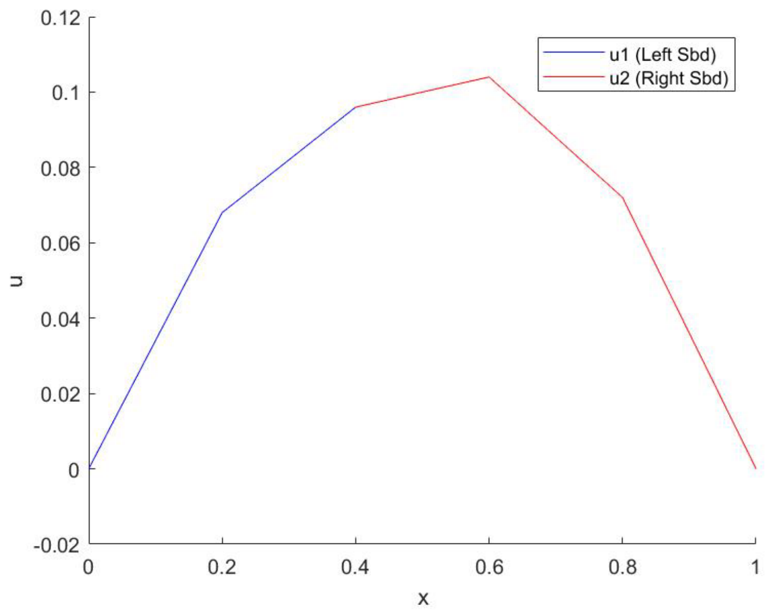
Exercise 8.2 – Aitken's Relaxation Scheme

MatLab code:

```
clc
close all
clear variables

% Problem 1
% Geometry
Data.inix    =    0;
Data.endx  = 0.4;
Data.nelem = 2;
% Physical properties
Data.kappa = 1;
Data.source = 1;
% Boundary conditions - Dirichlet
Data.FixLeft = 1; % 0: do not fix, 1: fix
Data.LeftValue = 0;
Data.FixRight = 0;
Data.RightValue = 0;

% Boundary conditions – Neumann
Data.FixFluxesLeft = 0;
Data.LeftFluxes = 0;
Data.FixFluxesRight = 1;
Data.RightFluxes = 0;

% Problem 2
% Geometry
Data2.inix = 0.4;
Data2.endx = 1;
Data2.nelem = 3;
% Physical properties
Data2.kappa = 1;
Data2.source = 1;
% Boundary conditions - Dirichlet
Data2.FixLeft = 1; % 0: do not fix, 1: fix
Data2.LeftValue = 0;
Data2.FixRight = 1;
Data2.RightValue = 0;
% Boundary conditions - Neumann
Data2.FixFluxesLeft = 0;
Data2.LeftFluxes = 0;
Data2.FixFluxesRight = 0;
Data2.RightFluxes = 0;

% Initialization
w = 1; % Initial relaxation
HeatProblem.Solution.uRight = 0; %Initial value at the interface = 0 - Sbd 1 at i
HeatProblem2.Solution.uLeft = 0; %Initial value at the interface = 0 - Sbd 2 at i
uGamma21_1 = 0; %Initial Sbd 2 interface value at i-1 / = 0
uGamma21_2 = 0; %Initial Sbd 2 interface value at i-2 / = 0
uGamma12_1 = 0; %Initial Sbd 1 interface value at i-1 / = 0

% Cycle data initialization
i = 1; % Iteration Counter initialization
imax = 100; % Maximum iterations
err = 100; % Error initialization
tol = 10^-6; % Tolerance
```

```
% Cycle "while"
 while (i < imax && err > tol)

    % Problem 1
    HeatProblem = HP_Initialize(Data); % Initialization
    HeatProblem = HP_Build(HeatProblem); % Building
    HeatProblem = HP_Solve(HeatProblem); % Solving

    % Calculation of Aitken relaxation (if iter counter is > 2)
    if i > 2
        w = (uGamma21_2 - uGamma21_1)/(uGamma21_2 - uGamma21_1 + HeatProblem.Solution.uRight -
    uGamma12_1);
    end

    % Calculation of uGamma21 at the current iteration
    Data2.LeftValue = w*HeatProblem.Solution.uRight + (1-w)*uGamma21_1;

    % Updating the interface values from the Sbd 2
    uGamma21_2 = uGamma21_1; % the i-1 becomes i-2 for the following step
    uGamma21_1 = Data2.LeftValue; % the i becomes i-1 for the following step

    % Problem 2
    HeatProblem2 = HP_Initialize(Data2); % Initialization
    HeatProblem2 = HP_Build(HeatProblem2); % Building
    HeatProblem2 = HP_Solve(HeatProblem2); % Solving

    % Updating the flux going to Sbd 1 (for the following step)
    Data.RightFluxes = -HeatProblem2.Solution.FluxesLeft;

    % Error calculation
    err = abs(abs(HeatProblem.Solution.uRight -
    uGamma12_1)/uGamma12_1)*100;
    fprintf('The current error is: %f\n',err);

    % Recalculating the interface value (Sbd 1) for the following step (it will be i-1 in the next iteration)
    uGamma12_1 = HeatProblem.Solution.uRight;

    % Incrementation of the iteration index
    i = i + 1;
end

HP_Plot(HeatProblem,1,1); %Plot Problem1
HP_Plot(HeatProblem2,2,1); %Plot Problem2 (in the same figure)
legend('u1 (Left Sbd)','u2 (Right Sbd)','Location','northeast'); % Legend
```

Final graph:



Problem 3

Dividing the monolithic problem (1 domain) in linear elements with size $h = 0.25$, leads to a domain discretized into 4 elements and 5 nodes. The characteristic size of the global quantities will be 5×5 for matrices and 5×1 for vectors. The following problem is going to be solved (discretized in time using a BDF1 scheme):

$$\left(\frac{\underline{\underline{M}}}{\Delta t} + \underline{\underline{K}}\right)\underline{u}^{n+1} = \underline{f} + \underline{\underline{M}}\,\underline{u}^{n} \qquad (\text{hypothesis } \Delta t = 1)$$

$k = 1$

$f = 1$

The element stiffness matrix, as seen in problem **5.1** is the following:

$$\underline{\underline{K}}^{(e)} = \frac{\overset{k}{1}}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

The element mass matrix, instead, is:

$$\underline{\underline{M}}^{(e)} = h\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} = \begin{bmatrix} 1/12 & 1/24 \\ 1/24 & 1/12 \end{bmatrix}$$

Once assembled, we obtain:

1) $$\underline{\underline{K}} = \begin{bmatrix} 4 & -4 & 0 & 0 & 0 \\ -4 & 8 & -4 & 0 & 0 \\ 0 & -4 & 8 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix}$$

2) $$\underline{\underline{M}} = \begin{bmatrix} 1/12 & 1/24 & 0 & 0 & 0 \\ 1/24 & 1/6 & 1/24 & 0 & 0 \\ 0 & 1/24 & 1/6 & 1/24 & 0 \\ 0 & 0 & 1/24 & 1/6 & 1/24 \\ 0 & 0 & 0 & 1/24 & 1/12 \end{bmatrix}$$

Giving the boundary conditions using Lagrange multipliers:

$$\underline{\underline{A}}\,\underline{u} = \underline{b} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\underline{A}\ (\text{boolean Matrix})}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \underline{b}$$

The whole global problem can be stated as follows:

$$\tilde{\underline{\underline{A}}}\,\underline{U}^{n+1} = \tilde{\underline{f}}$$

where $\tilde{\underline{\underline{A}}} = \dfrac{\underline{\underline{M}}}{\Delta t} + \underline{\underline{K}}$ and $\tilde{\underline{f}} = \underline{f} + \dfrac{\underline{\underline{M}}}{\Delta t}\,\underline{U}^n$

So, the lagrange multiplier $\underline{\lambda}$ is added to the global problem in this way:

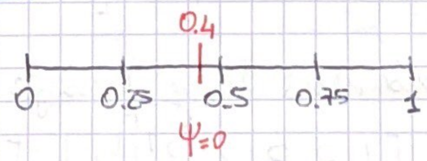
$$\begin{bmatrix} \tilde{\underline{\underline{A}}} & \underline{\underline{A}}^T \\ \underline{\underline{A}} & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{U}^{n+1} \\ \underline{\lambda} \end{bmatrix} = \begin{bmatrix} \tilde{\underline{f}} \\ \underline{b} \end{bmatrix}$$

$\underline{\lambda}$ describes the reaction on the Dirichlet boundary. The matrix $\begin{bmatrix} \tilde{\underline{\underline{A}}} & \underline{\underline{A}}^T \\ \underline{\underline{A}} & \underline{\underline{0}} \end{bmatrix}$ will be called $\underline{\underline{B}}$. Here $\Delta t = 1$ for simplicity

If we perform (for example in Matlab) the condition number of $\underline{\underline{B}}$ we obtain $k_B = \|\underline{\underline{B}}\|\cdot\|\underline{\underline{B}}^{-1}\| = 38.3156$ which is almost half of the condition number for the original problem $\left(k_{\tilde{A}} = \|\tilde{\underline{\underline{A}}}\|\cdot\|\tilde{\underline{\underline{A}}}^{-1}\|\right)$ $k_{\tilde{A}} = 73.0412$.

We can see that the use of extra conditions (B.C.$_{\text{Dirichlet}}$ seen using the Lagrangian Multiplier scheme) increases the number of equations but makes the condition number smaller which consists in a much cheaper computational cost.

Problem 4



Again the global problem (system of equations) is the following:

$$(\underline{\underline{M}} + \underline{\underline{K}})\,\underline{U}^{n+1} = \underline{f} + \underline{\underline{M}}\,\underline{U}^n \quad (\Delta t = 1, \text{ see before})$$

The part which remains untouched of the stiffness matrix is the contribution from the elements 3 and 4. The element 1 differs for a higher value of the diffussivity: $k = 100$. It's obvious the charge:

$$\underline{\underline{K}}^{(1)} = 100\, \underline{\underline{K}}^{(1)}_{previous} = \begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix}$$

As for the second element, the calculation is a bit complex: in fact we have:

$$K^{(2)}_{11} = (K^{(2)}_{22}) = 1 \cdot \int_{0.25}^{0.4} \left(\frac{(-1)x}{h}\right)\left(\frac{(-)x}{h}\right) dx + 100 \int_{0.4}^{0.5} \left(\frac{(-)x}{h}\right)\left(\frac{(-)x}{h}\right) dx = 242.4$$

$$K^{(2)}_{12} = K^{(2)}_{21} = 1 \cdot \int_{0.25}^{0.4} \left(\frac{x}{h}\right)\left(-\frac{x}{h}\right) dx + 100 \int_{0.4}^{0.5} \left(\frac{x}{h}\right)\left(-\frac{x}{h}\right) dx = -242.4$$

$$\underline{\underline{K}}^{(2)} = \begin{bmatrix} 242.4 & -242.4 \\ -242.4 & 242.4 \end{bmatrix}$$

The assembled matrix will be the following:

$$\underline{\underline{K}} = \begin{bmatrix} 400 & -400 & 0 & 0 & 0 \\ -400 & 642.2 & -242.4 & 0 & 0 \\ 0 & -242.4 & 246.4 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix}$$

while the mass matrix is the same as before:

$$\underline{\underline{M}} = \begin{bmatrix} 1/12 & 1/24 & 0 & 0 & 0 \\ 1/24 & 1/6 & 1/24 & 0 & 0 \\ 0 & 1/24 & 1/6 & 1/24 & 0 \\ 0 & 0 & 1/24 & 1/6 & 1/24 \\ 0 & 0 & 0 & 1/24 & 1/12 \end{bmatrix}$$