Universitat Politecnica de Catalunya CIMNE
Master in Computational Mechanics

Computational Solid Mechanics

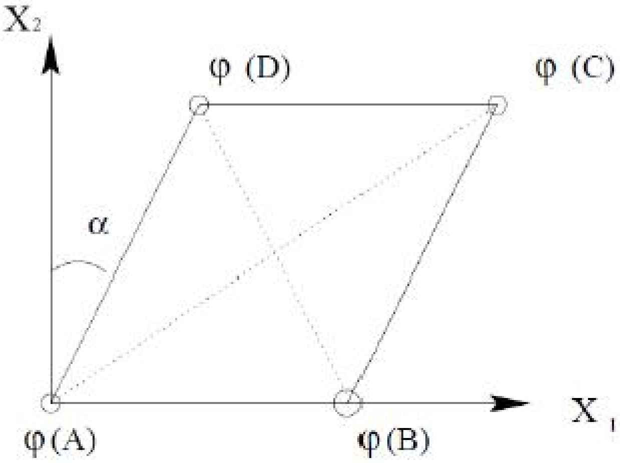
Homeworks 1b & 1c

Part 3: Non Linear

Submitted By: Kiran Kolhe

28th April 2020

Homework 1b :



A two-dimensional solid is contained in the $\{X1, X2\}$ coordinate plane relative to an orthonormal cartesian basis $\{E_I\}$, I = 1, 2, 3. The solid is initially square in shape and is enclosed in a rigid truss frame hinged at the corners A, B, C, and D of the square, so that the sides AB, BC, CD and DA cannot change their length. The deformation is presumed homogeneous and is parametrized by the angle α rotated by the sides DA and BC.

Solution:

1) Write the deformation mapping in terms of α.

The deformation map is given by, x = φ (X, t) = φ $\begin{bmatrix} X_1 + X_2 \sin\alpha \\ X_2 \cos\alpha \end{bmatrix}$

2) Compute the deformation gradient *F* and the right Cauchy-Green deformation tensor *C*.

So the value of deformation gradient becomes,

$$F = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} \end{bmatrix} = \begin{bmatrix} 1 & sin\,\alpha \\ 0 & cos\,\alpha \end{bmatrix}$$

The Right Cauchy Green deformation tensor is given by, $C = F^T * F$

$$C = \begin{bmatrix} 1 & sin\,\alpha \\ sin\,\alpha & sin^2(\alpha) + cos^2(\alpha) \end{bmatrix}$$

3) Compute and plot the variation in volume of the solid as a function of α.

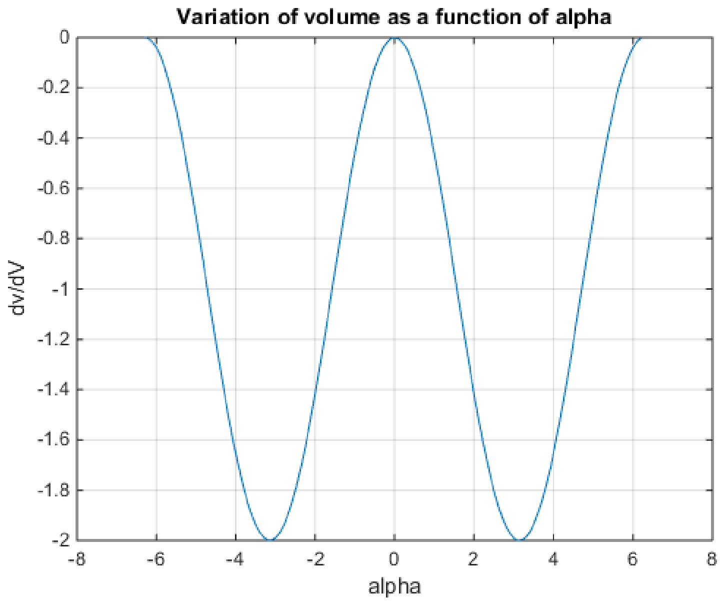
The variation of volume is given by the relation dv = J dV.
The Jacobian 'J' is defined as, J = det F.
$\therefore J = cos\,\alpha$
The deformation is homogeneous and the variation can be given by

$$\therefore \frac{dv - dV}{dV} = \frac{dv}{dV} - 1 = J - 1 = cos\,\alpha - 1$$

Substituting in the volume relation,

$$\frac{dv}{dV} = cos\,\alpha - 1$$

The plot of this relation is shown below.



Variation of $\frac{dv}{dV}$ from alpha -2π to 2π

4) At what point do the deformations cease to be admissible? Interpret geometrically.

For the deformation to exist J > 0 always. Hence, the deformations cease to be admissible when J < 0.
We know that, $J = cos\,\alpha$
$\therefore cos\,\alpha < 0$ should be maintained.
$\therefore \alpha > 90°$ is the condition for the deformation to be not admissible.

5) Compute the change in length of the diagonals AC and BD, and the change in the angle β subtended by them. Interpret geometrically. Plot the change of lengths and the change of angle β as a function of α.

The length variation of diagonal AC is given by $\lambda_{AC} = \frac{AC_{final}}{AC_{initial}}$.

But, we also know that, $\lambda_{AC}^2 = {N_{AC}}^T * C * N_{AC}$

$$\therefore \lambda^2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1 & \sin\alpha \\ \sin\alpha & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ 1\end{bmatrix}$$

$$\lambda_{AC} = \sqrt{1+\sin\alpha}$$

$$AC_{final} = \left(\sqrt{1+\sin\alpha}\right) AC_{initial}$$

Also, for diagonal BD $\lambda_{BD} = \frac{BD_{final}}{BD_{initial}}$

$$\lambda_{BD}^2 = {N_{BD}}^T * C * N_{BD}$$

$$\therefore \lambda^2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}1 & \sin\alpha \\ \sin\alpha & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}-1 \\ 1\end{bmatrix}$$

$$\lambda_{BD} = \sqrt{1-\sin\alpha}$$

$$BD_{final} = \left(\sqrt{1-\sin\alpha}\right) BD_{initial}$$

The final lengths after deformation of the diagonals AC & BD vary as $1+\sin\alpha$ & $1-\sin\alpha$ respectively times the initial lengths before deformation.

The change in β which is the angle between the diagonals can be interpreted as,

$$\cos\beta = \frac{N_{AC} * (1+2E) * N_{BD}}{\sqrt{1+2N_{AC} * E * N_{AC}} * \sqrt{1+2N_{BD} * E * N_{BD}}}$$

E is the Green Lagrange strain tensor given by, $E = \frac{1}{2}(C-1)$

$$\therefore E = \frac{1}{2}\left\{\begin{bmatrix}1 & \sin\alpha \\ \sin\alpha & 1\end{bmatrix} - \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\right\}$$

$$\therefore E = \frac{1}{2}\begin{bmatrix}0 & \sin\alpha \\ \sin\alpha & 0\end{bmatrix}$$

Now,

$$\sqrt{1+2N_{AC} * E * N_{AC}} = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} + 2 * \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1 & \sin\alpha \\ \sin\alpha & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ 1\end{bmatrix} = \sqrt{1+\sin\alpha}$$

$$\sqrt{1+2N_{BD} * E * N_{BD}} = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} + 2 * \frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}1 & \sin\alpha \\ \sin\alpha & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}-1 \\ 1\end{bmatrix} = \sqrt{1-\sin\alpha}$$

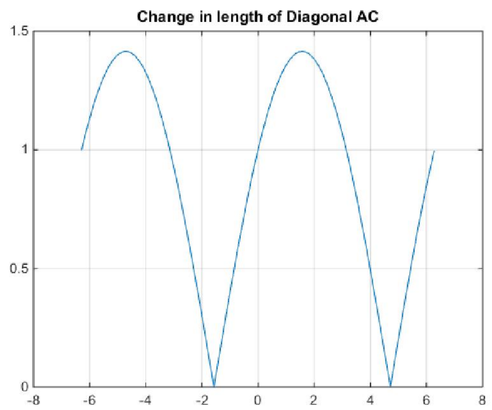
So,

$$\cos\beta = \frac{\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix} * \left(\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} + \frac{1}{2} * 2\begin{bmatrix}0 & \sin\alpha \\ \sin\alpha & 0\end{bmatrix}\right) * \frac{1}{\sqrt{2}}\begin{bmatrix}-1 \\ 1\end{bmatrix}}{\sqrt{1+\sin\alpha} * \sqrt{1-\sin\alpha}}$$

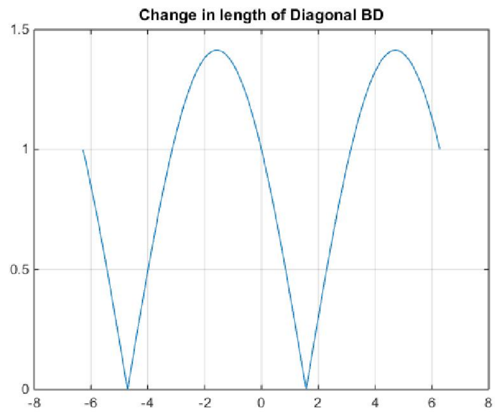
$$\therefore \cos\beta = \frac{0}{1-\sin\alpha} = 0$$

$$\therefore \beta = 90°$$

So we observe that the angle does not change even after deformation.



Variation of Diagonal AC



Variation of Diagonal BD

Homework 1c :

Consider a cylindrical solid referred to an orthonormal Cartesian reference frame {$X1$, $X2$, $X3$}, whose axis aligned with the $X3$ direction. Its normal cross section occupies a region Ω in the {$X1$, $X2$} plane of boundary $\partial\Omega$. An anti-plane shear deformation of the solid can be defined as one for which the deformation mapping is of the form:

$$\varphi 1 = X^1,\ \varphi 2 = X^2,\ \varphi 3 = X^3 + w\ (X^1, X^2).$$

The spatial and material reference frames are taken to coincide, and the function w is defined over Ω.

Solution:

1) Sketch the deformation of the region Ω.

a) Compute the deformation gradient field *F*, the right Cauchy-Green deformation tensor *C*, and the Jacobian *J* of the deformation field in terms of *w*.

The deformation map is given by, $x = \varphi\ (X) = \varphi \begin{bmatrix} X^1 \\ X^2 \\ X^3 + w(X^1, X^2) \end{bmatrix}$

The value of deformation gradient becomes,

$$F = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_2} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 1 \end{bmatrix}$$

The Right Cauchy Green deformation tensor is given by, $C = F^T * F$

$$C = \begin{bmatrix} 1 + \left(\frac{\partial w}{\partial X_1}\right)^2 & \frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2} & \frac{\partial w}{\partial X_1} \\ \frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2} & 1 + \left(\frac{\partial w}{\partial X_2}\right)^2 & \frac{\partial w}{\partial X_2} \\ \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 1 \end{bmatrix}$$

The Jacobian is given by J = det(F),

$$\therefore J = \left(1 + \left(\frac{\partial w}{\partial X_1}\right)^2\right)\left\{\left(1 + \left(\frac{\partial w}{\partial X_2}\right)^2\right) - \left(\frac{\partial w}{\partial X_2}\right)^2\right\} - \frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2}\left\{\left(\frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2}\right) - \frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2}\right\}$$
$$+ \frac{\partial w}{\partial X_1}\left\{\left(\frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2}\right)\frac{\partial w}{\partial X_2} - \left(1 + \left(\frac{\partial w}{\partial X_2}\right)^2\right)\frac{\partial w}{\partial X_1}\right\}$$

$$\therefore J = 1$$

b) Does the solid change volume during the deformation?

As J = 1, from the relation dv = J dV we know that the initial and final volume are the same. Hence, the volume does not change during the deformation.

c) Are the local impenetrability conditions satisfied?

As J > 0, the impenetrability conditions are satisfied.

.

2) Consider the unit vectors:

$$\boldsymbol{A} = \frac{\frac{\partial w}{\partial X_1}\boldsymbol{E_1} + \frac{\partial w}{\partial X_2}\boldsymbol{E_2}}{\sqrt{\left(\frac{\partial w}{\partial X_1}\right)^2 + \left(\frac{\partial w}{\partial X_2}\right)^2}} \quad and \quad \boldsymbol{B} = \frac{-\frac{\partial w}{\partial X_1}\boldsymbol{E_1} + \frac{\partial w}{\partial X_2}\boldsymbol{E_2}}{\sqrt{\left(\frac{\partial w}{\partial X_1}\right)^2 + \left(\frac{\partial w}{\partial X_2}\right)^2}}$$

where {EI}, I = 1, 2, 3 are the (orthonormal) material basis vectors.

a) How are $\boldsymbol{A}$ and $\boldsymbol{B}$ related to the level contours of w ($X1, X2$)?

The gradient of level contour w ($X1, X2$) is given by,

$$\nabla w = \frac{\partial w}{\partial X_1}\boldsymbol{E_1} + \frac{\partial w}{\partial X_2}\boldsymbol{E_2}$$

Now, if we calculate the unit vector for w, it becomes,

$$\widehat{w} = \frac{\frac{\partial w}{\partial X_1}\boldsymbol{E_1} + \frac{\partial w}{\partial X_2}\boldsymbol{E_2}}{\sqrt{\left(\frac{\partial w}{\partial X_1}\right)^2 + \left(\frac{\partial w}{\partial X_2}\right)^2}}$$

which is the same as the vector A. Hence, we can say that **A** is the unit normal vector to level contours of w ($X1, X2$), and since **A** and **B** are perpendicular to each other, we can also say that **B** is unit tangent vector of w ($X1, X2$).

b) Compute (in terms of w) **the change in length** (measured by the corresponding stretch ratios) **of $\boldsymbol{A}$ and $\boldsymbol{B}$, as well as the change in the angle subtended by $\boldsymbol{A}$ and $\boldsymbol{B}$.**

The change in length can be calculated using the stretch relation given by,

$$\lambda_A^2 = N_A{}^T * C * N_A = 1 + 2N_A{}^T E N_A$$

$$\lambda_A^2 = \frac{\frac{\partial w}{\partial X_1}\boldsymbol{E_1} + \frac{\partial w}{\partial X_2}\boldsymbol{E_2}}{\sqrt{\left(\frac{\partial w}{\partial X_1}\right)^2 + \left(\frac{\partial w}{\partial X_2}\right)^2}} \begin{bmatrix} 1 + \left(\frac{\partial w}{\partial X_1}\right)^2 & \frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2} & \frac{\partial w}{\partial X_1} \\ \frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2} & 1 + \left(\frac{\partial w}{\partial X_2}\right)^2 & \frac{\partial w}{\partial X_2} \\ \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 1 \end{bmatrix} \frac{\frac{\partial w}{\partial X_1}\boldsymbol{E_1} + \frac{\partial w}{\partial X_2}\boldsymbol{E_2}}{\sqrt{\left(\frac{\partial w}{\partial X_1}\right)^2 + \left(\frac{\partial w}{\partial X_2}\right)^2}}$$

$$\therefore \lambda_A^2 = \frac{1}{\sqrt{\left(\frac{\partial w}{\partial X_1}\right)^2 + \left(\frac{\partial w}{\partial X_2}\right)^2}} \left\{ \begin{bmatrix} \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 0 \end{bmatrix} \begin{bmatrix} 1 + \left(\frac{\partial w}{\partial X_1}\right)^2 & \frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2} & \frac{\partial w}{\partial X_1} \\ \frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2} & 1 + \left(\frac{\partial w}{\partial X_2}\right)^2 & \frac{\partial w}{\partial X_2} \\ \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial X_1} \\ \frac{\partial w}{\partial X_2} \\ 1 \end{bmatrix} \right\}$$

$$\therefore \lambda_A = \sqrt{1 + \left(\frac{\partial w}{\partial X_1}\right)^2 + \left(\frac{\partial w}{\partial X_2}\right)^2}$$

Similarly, the stretch ratio for vector **B** can be calculated as above, and it is,

$$\lambda_B^2 = \frac{1}{\sqrt{\left(\frac{\partial w}{\partial X_1}\right)^2 + \left(\frac{\partial w}{\partial X_2}\right)^2}} \left\{ \begin{bmatrix} -\frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 0 \end{bmatrix} \begin{bmatrix} 1 + \left(\frac{\partial w}{\partial X_1}\right)^2 & \frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2} & \frac{\partial w}{\partial X_1} \\ \frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2} & 1 + \left(\frac{\partial w}{\partial X_2}\right)^2 & \frac{\partial w}{\partial X_2} \\ \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 1 \end{bmatrix} \begin{bmatrix} -\frac{\partial w}{\partial X_1} \\ \frac{\partial w}{\partial X_2} \\ 1 \end{bmatrix} \right\}$$

$$\therefore \lambda_B = 1$$

The change in angle subtended by these 2 vectors can be calculated as,

$$\cos\theta = \frac{{N_A}^T * (1 + 2\boldsymbol{E})N_B}{\sqrt{1 + 2{N_A}^T E N_A}\sqrt{1 + 2{N_B}^T E N_B}}$$

$$\therefore \cos\theta = \frac{{N_A}^T * (C)N_B}{\lambda_A^2 * \lambda_B^2}$$

$$\therefore \cos\theta = \frac{\frac{1}{\sqrt{\left(\frac{\partial w}{\partial X_1}\right)^2 + \left(\frac{\partial w}{\partial X_2}\right)^2}} \left\{ \begin{bmatrix} \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 0 \end{bmatrix} \begin{bmatrix} 1 + \left(\frac{\partial w}{\partial X_1}\right)^2 & \frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2} & \frac{\partial w}{\partial X_1} \\ \frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2} & 1 + \left(\frac{\partial w}{\partial X_2}\right)^2 & \frac{\partial w}{\partial X_2} \\ \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 1 \end{bmatrix} \begin{bmatrix} -\frac{\partial w}{\partial X_1} \\ \frac{\partial w}{\partial X_2} \\ 1 \end{bmatrix} \right\}}{\left(\sqrt{1 + \left(\frac{\partial w}{\partial X_1}\right)^2 + \left(\frac{\partial w}{\partial X_2}\right)^2}\right) * (1)}$$

$$\therefore \cos\theta = \frac{\mathbf{0}}{\left(\sqrt{1 + \left(\frac{\partial w}{\partial X_1}\right)^2 + \left(\frac{\partial w}{\partial X_2}\right)^2}\right)}$$

$$\therefore \theta = \frac{\pi}{2}$$

c) Interpret the results.

- Vector **A** is the unit normal vector and vector **B** is the tangent vector to the level contours of $w\,(X1, X2)$.
- Vector **A** undergoes a stretch of $\sqrt{1 + \left(\frac{\partial w}{\partial X_1}\right)^2 + \left(\frac{\partial w}{\partial X_2}\right)^2}$ while vector **B** does not undergo any stretch.
- Both the vectors remain orthonormal only before and after deformation.

3) Using the Piola transformation, compute (in terms of w) **the change in area of, and in the normal to, an infinitesimal material area contained in the $\{X1, X2\}$ plane.**

Using Piola transform, we have the relation,

$$d\boldsymbol{a} = J\boldsymbol{F}^{-T} d\boldsymbol{A} \quad \rightarrow \quad da\boldsymbol{n} = J\boldsymbol{F}^{-T} dA\boldsymbol{N}$$

Where **n** and **N** are the unit vector to the area da and dA

Here, we know that,

$J = 1$ and $\boldsymbol{F}^{-T} = \begin{bmatrix} 1 & 0 & -\frac{\partial w}{\partial X_1} \\ 0 & 1 & -\frac{\partial w}{\partial X_2} \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Thus, the change in area can be written as,

$$da = 1 * \begin{bmatrix} 1 & 0 & -\frac{\partial w}{\partial X_1} \\ 0 & 1 & -\frac{\partial w}{\partial X_2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dA$$

$$\therefore da = \begin{bmatrix} -\frac{\partial w}{\partial X_1} \\ -\frac{\partial w}{\partial X_2} \\ 1 \end{bmatrix} dA$$

5) Derive an integral expression for the deformed area of the domain Ω.

Integrating the relation $da\boldsymbol{n} = J\boldsymbol{F}^{-T} dA\boldsymbol{N}$, we get

$$\int_{\Omega} da\, \boldsymbol{n} = \int_{\Omega} J\boldsymbol{F}^{-T}\boldsymbol{N} dA\, d\Omega$$

Where $\boldsymbol{n} = \dfrac{\begin{bmatrix} -\frac{\partial w}{\partial X_1} \\ -\frac{\partial w}{\partial X_2} \\ 1 \end{bmatrix}}{\sqrt{1+\left(\frac{\partial w}{\partial X_1}\right)^2+\left(\frac{\partial w}{\partial X_2}\right)^2}}$

$$\therefore \int_{\Omega} \frac{\begin{bmatrix} -\frac{\partial w}{\partial X_1} \\ -\frac{\partial w}{\partial X_2} \\ 1 \end{bmatrix}}{\sqrt{1 + \left(\frac{\partial w}{\partial X_1}\right)^2 + \left(\frac{\partial w}{\partial X_2}\right)^2}} da = \int_{\Omega} \begin{bmatrix} -\frac{\partial w}{\partial X_1} \\ -\frac{\partial w}{\partial X_2} \\ 1 \end{bmatrix} dA\, d\Omega$$

$$\therefore da = \sqrt{1 + \left(\frac{\partial w}{\partial X_1}\right)^2 + \left(\frac{\partial w}{\partial X_2}\right)^2}\, d\Omega$$

6) Let the boundary $\partial\Omega$ of Ω be defined parametrically by the equations

$$X_1 = X_1(S), X_2 = X_2(S)$$

where $0 \leq S \leq L$ is the arc-length measured along $\partial\Omega$. Note that $\boldsymbol{E}1X1(S)/dS + \boldsymbol{E}2X2(S)/dS$ is the unit vector tangent to $\partial\Omega$. Derive an integral expression for the perimeter of the deformed boundary $\varphi(\partial\Omega)$.

We know that, length of the curve is given by ,

$$l = \int ds = \int \lambda \, dS = \int \sqrt{1 + 2\mathrm{T}E\mathrm{T}}$$

Where, $T = \left(\frac{\mathrm{X}_1(\mathrm{S})}{dS}, \frac{\mathrm{X}_2(\mathrm{S})}{dS}, 0\right)$

$$l = \int_\Omega \sqrt{1 + 2T \cdot ET} dS = \sqrt{1 + 2\left(\frac{\mathrm{X}_1(\mathrm{S})}{dS}, \frac{\mathrm{X}_2(\mathrm{S})}{dS}, 0\right) * \frac{1}{2}\begin{bmatrix} \left(\frac{\partial w}{\partial X_1}\right)^2 & \frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2} & \frac{\partial w}{\partial X_1} \\ \frac{\partial w}{\partial X_1}\frac{\partial w}{\partial X_2} & \left(\frac{\partial w}{\partial X_2}\right)^2 & \frac{\partial w}{\partial X_2} \\ \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 0 \end{bmatrix}\begin{Bmatrix} \frac{\mathrm{X}_1(\mathrm{S})}{dS} \\ \frac{\mathrm{X}_2(\mathrm{S})}{dS} \\ 0 \end{Bmatrix}}$$

$$\therefore l = \int_\Omega \sqrt{\left(\frac{\mathrm{X}_1(\mathrm{S})}{dS} w_{,1} + \frac{\mathrm{X}_2(\mathrm{S})}{dS} w_{,2}\right)^2 + 1} \; dS$$