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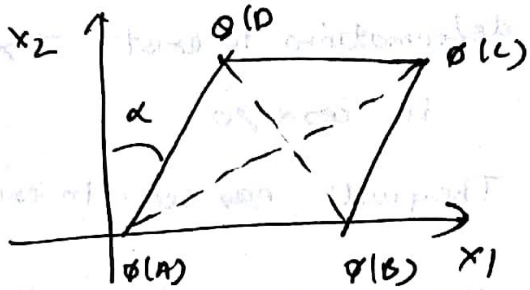
Computational Solid Mechanics



HW 1B

Solution

1) Deformation mapping in terms of α

The deformation map is given by $x = \phi(X, t) = \begin{bmatrix} X_1 + X_2 \sin\alpha \\ X_2 \cos\alpha \end{bmatrix}$

as the movement in X_1 results in deformation in x_1 but a movement along AD would be resolved in components.

Q2) Compute the deformation gradient & right cauchy green deformation Tensor

$$F = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} \end{bmatrix} = \begin{bmatrix} 1 & \sin\alpha \\ 0 & \cos\alpha \end{bmatrix}$$

$$C = F^T F = \begin{bmatrix} 1 & \sin\alpha \\ \sin\alpha & 1 \end{bmatrix}$$

Q3) Compute the variation in the volume of solid as a function of α

$dv = J dV$ where $J = \det(F) = \cos\alpha$

Volumetric deformation is (e) is denoted as $e = \dfrac{dv - dV}{dV} = J - 1$

$$e = \cos\alpha - 1$$

Q4 At what point do the deformations cease to be admissible

For deformations to exist $J > 0$

ie $\cos\alpha > 0$

They will ~~cea~~ cease to exist when $\cos\alpha < 0$ or when $\alpha > \frac{\pi}{2}$

geometrically, $\alpha > \frac{\pi}{2}$ would mean that the diagonals lie ~~or~~ above & below the x_1 axis. In such a case the area is negative.

Q5 Compute the change in length of the diagonals AC & BD and the change in the angle β subtended by them. Interpret geometrically. Plot the change of lengths and the change of angle β as a function of α.

– The length variation of diagonals be given by λ_{AC} & λ_{BD}

$$\lambda_{AC}^2 = N_{AC}^T \, C \, N_{AC}$$

$$\lambda_{AC}^2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \sin\alpha \\ \sin\alpha & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_{AC}^2 = 1 + \sin(\alpha)$$

$$\lambda_{AC} = \sqrt{1 + \sin\alpha}$$

similarly

$$\lambda_{BD}^2 = \frac{1}{2}\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \sin\alpha \\ \sin\alpha & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= 1 - \sin\alpha$$

$$\lambda_{BD} = \sqrt{1 - \sin\alpha}$$

The change in angle β is given by

$$\cos\beta = \frac{N_{AC}\,(1+2E)\,N_{BD}}{\sqrt{1+2N_{AC}EN_{AC}}\cdot\sqrt{1+2N_{BD}EN_{BD}}} \quad —①$$

$$E = \frac{1}{2}(C-I) = \frac{1}{2}\begin{bmatrix} 0 & \sin\alpha \\ \sin\alpha & 0 \end{bmatrix}$$

The denominators of eqn ① are already computed in the previous section to be $\sqrt{1+\sin\alpha}$ & $\sqrt{1-\sin\alpha}$

The numerator is given by

$$\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\cdot\begin{bmatrix} 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & \sin\alpha \\ \sin\alpha & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}\left(-1 + \sin\alpha - \sin\alpha + 1\right)$$

$$= 0$$

$$\therefore \cos\beta = 0$$

$$\beta = \frac{\pi}{2}$$

This is the original value of β $\therefore$ There is no change in the angle even after deformation

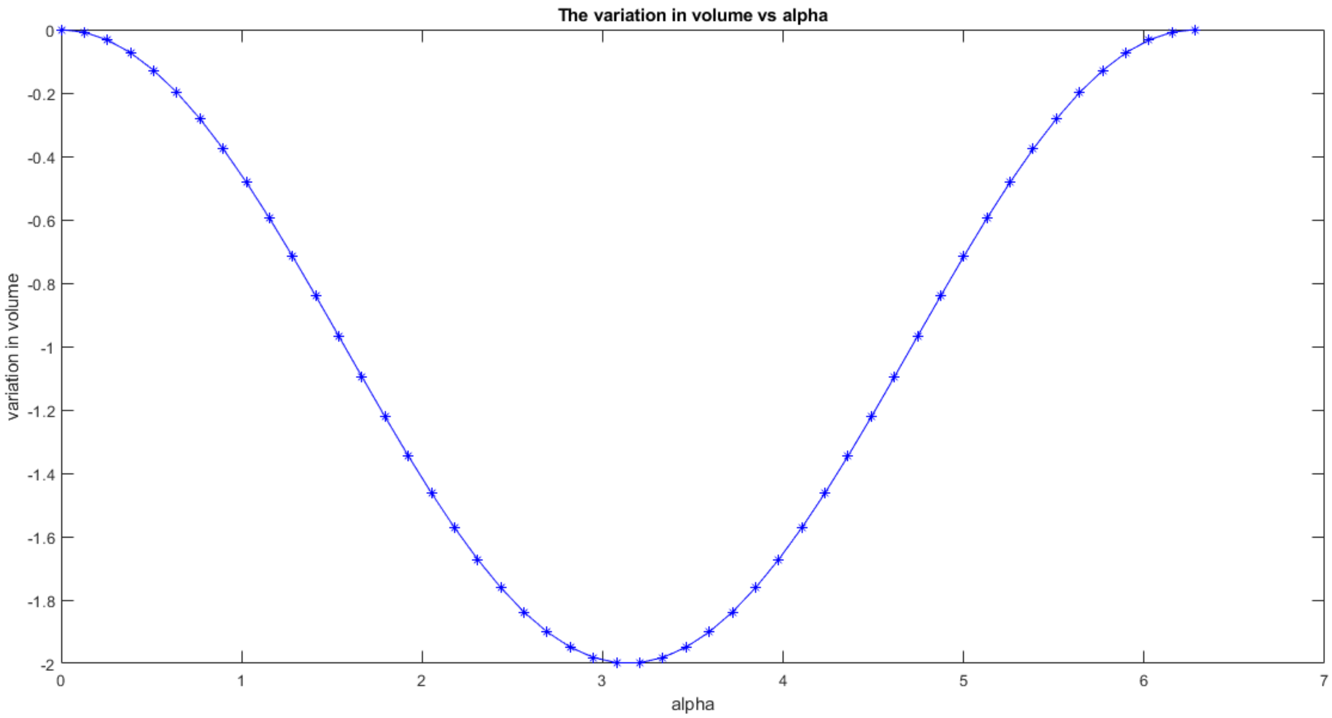


Fig for Q3: Plot of variation of Volume vs alpha

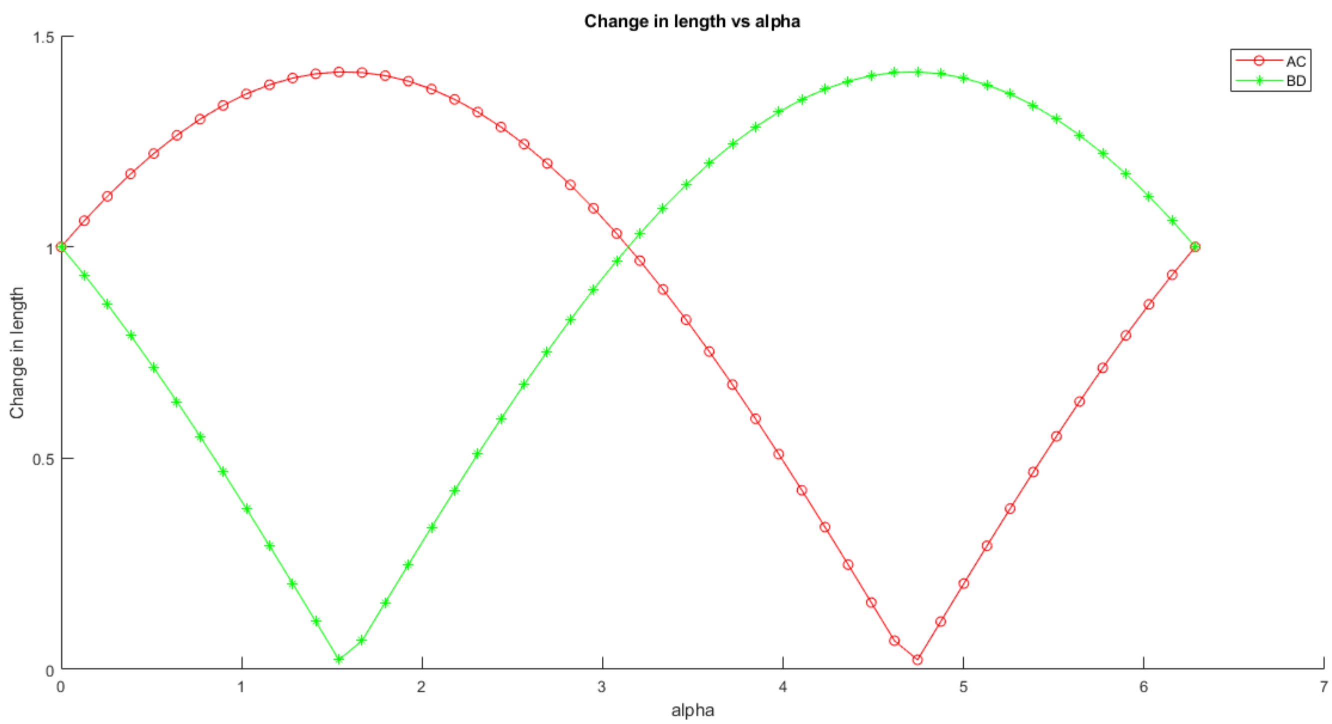


Figure for question 5: Plot of change in the length of diagonals vs alpha

HW 1c

$\phi_1 = x^1 \qquad \phi_2 = x^2 \qquad \phi_3 = x^3 + w(x_1, x_2)$

1) a)

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial w}{\partial x_1} & \frac{\partial w}{\partial x_2} & 1 \end{bmatrix}$$

$$C = F^T F = \begin{bmatrix} 1+\left(\frac{\partial w}{\partial x_1}\right)^2 & \frac{\partial w}{\partial x_1}\frac{\partial w}{\partial x_2} & \frac{\partial w}{\partial x_1} \\ \frac{\partial w}{\partial x_2}\frac{\partial w}{\partial x_1} & 1+\left(\frac{\partial w}{\partial x_2}\right)^2 & \frac{\partial w}{\partial x_2} \\ \frac{\partial w}{\partial x_1} & \frac{\partial w}{\partial x_2} & 1 \end{bmatrix}$$

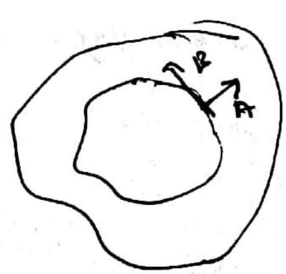
$$J = \det F = 1$$

Q1b Since the ~~det~~ variation in volume is given by $\rho = J \rho_0$ and $J = 1$ the solid does not change volume

Q1c Since $J > 0$, the local impenetrability conditions are satisfied

Q2 a We can see that $\bar{A} \cdot \bar{B} = 0$ $\therefore$ A & B must be perpendicular and in the same plane ~~A~~ also

$A = \dfrac{\nabla \cdot w}{\|\nabla \cdot w\|}$ $\therefore$ A is the unit normal vector to the contour & B must be the unit tangent vector



. The angle subtended by A & B is $\frac{\pi}{2}$

$w_{,i}$ is simply written as w_1 & $w_{,2} = w_2$

Q2b

$$\lambda_A^2 = N_A^T C N_A$$

$$= \begin{bmatrix} \frac{w_1}{\sqrt{w_1^2+w_2^2}} \\ \frac{w_2}{\sqrt{w_1^2+w_2^2}} \\ 0 \end{bmatrix}^T \begin{bmatrix} 1+w_1^2 & w_1 w_2 & w_1 \\ w_1 w_2 & 1+w_2^2 & w_2 \\ w_1 & w_2 & 1 \end{bmatrix} \begin{bmatrix} \frac{w_1}{\sqrt{w_1^2+w_2^2}} \\ \frac{w_2}{\sqrt{w_1^2+w_2^2}} \\ 0 \end{bmatrix}$$

$$\lambda_A^2 = (w_1^2 + w_2^2 + 1) \quad = \therefore \lambda_A = \sqrt{\left(\frac{\partial w}{\partial x_1}\right)^2 + \left(\frac{\partial w}{\partial x_2}\right)^2 + 1}$$

$$\lambda_B^2 = \begin{bmatrix} \frac{-w_2}{\sqrt{w_1^2+w_2^2}} \\ \frac{w_1}{\sqrt{w_1^2+w_2^2}} \\ 0 \end{bmatrix}^T \begin{bmatrix} 1+w_1^2 & w_1 w_2 & w_1 \\ w_1 w_2 & (1+w_2^2) & w_2 \\ w_1 & w_2 & 1 \end{bmatrix} \begin{bmatrix} \frac{-w_2}{\sqrt{w_1^2+w_2^2}} \\ \frac{w_1}{\sqrt{w_1^2+w_2^2}} \\ 0 \end{bmatrix}$$

$$\lambda_B^2 = 1 \quad \therefore \lambda_B = 1$$

The angle subtended is $\frac{\pi}{2}$.

Q2c The vector A increases in size as the ~~stretch~~ stretch ratio is positive & greater than 1 but & the basis vector B remains of the same size.

Q3

Let dS be an infintesimal area in the reference configuration
A unit vector normal to such an area can be defined as $[0\ 0\ 1]$

$d\bar{s} = J F^{-T} d\bar{S}$ according to the Nanson's Formula

$\therefore$ ~~dsni = +.~~

$$F^{-T} = \begin{bmatrix} 1 & 0 & -w_1 \\ 0 & 1 & -w_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\text{\&}\quad ds\,n_i = [0 \quad 0 \quad -(w_1+w_2)+1]$

OR $[0 \quad 0 \quad -(\frac{\partial w}{\partial x_1} + \frac{\partial w}{\partial x_2}) + 1]$

Q4

Q5 Desire an integral expression for the deformed area of the domain Ω

using the relation ~~ΔS~~,
$\Delta s\, n = \det(F)\, F^{-T} N\, dS$

To compute the deformed area we get

$(\Delta S)\pi \cdot (\Delta s) n = \det(F)\,(F^{-T})^2\, N\,(\Delta A)$

$$a = \int_A \det(F) \sqrt{C^{-1} N \cdot N}\; dA$$

here $N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ & $C^{-1} = \begin{bmatrix} 1 & 0 & -w_1 \\ 0 & 1 & -w_2 \\ -w_1 & -w_2 & w_1^2 + w_2^2 + 1 \end{bmatrix}$

∴ The integral expression reduces to

$$\boxed{S = \int_A \sqrt{w_1^2 + w_2^2 + 1}\, dA}$$

Q6 The boundary $\partial \Omega$ of Ω is parametrically defined by

$x^1 = x^1(s)$ & $x^2 = x^2(s)$ $0 \le s \le L$ is the arc length

& $\frac{x^1(s)}{ds} . \bar{E}_1 + \bar{E}_2 \frac{x^2(s)}{ds}$ is the vector tangent to $\partial \Omega$

→ The deformed length of the curve is given by

$$\Delta L = \int_0^l \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2 + \left(\frac{dz}{d\xi}\right)^2}\, d\xi$$ here ξ is the parameter

In this particular case it can be rewritten as

$$\Delta l = \int_0^l \sqrt{C A . A \left[\left(\frac{dx^1}{ds}\right)^2 + \left(\frac{dx^2}{ds}\right)^2 \right]}\, ds$$

& $A = \begin{bmatrix} \frac{x_1}{ds} \\ \frac{x_2}{ds} \\ 0 \end{bmatrix}$

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$$\therefore \quad CA \cdot A = \frac{(1+w_{,1}{}^{2})x_1{}^2 + 2w_1w_2x_1x_2 + (1+w_{,2}{}^{2})x_2{}^2}{ds^2}$$

$$\Delta L = \int_0^{l} \sqrt{(1+w_{,1})^2 x_1{}^2 + 2w_1w_2x_1x_2 + (1+w_{,2}{}^{2})x_2{}^2 \left(\left(\frac{dx_1}{ds}\right)^2 + \left(\frac{dx_2}{dx}\right)^2\right.} \quad \text{— (11)}$$

Eq (11) is the integral expression for the perimeter of the boundary.