CoSM - Part III Homework 1

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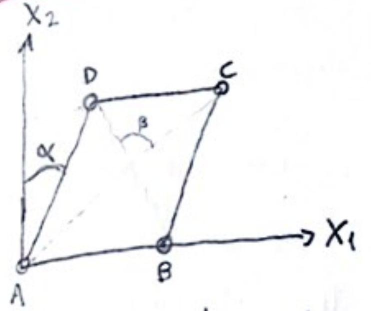
Exercise 2



2D solid contained in $\{X_1, X_2\}$ plane relative to an orthonormal cartesian basis $\{E_I\}$ $I = 1,2,3$. The solid is initially square and is enclosed in a rigid truss frame hinged at the corners A, B, C and D so that the sides cannot change their length. The deformation is presumed homogeneous and is parametrized by the angle α.

1.) Write the deformation mapping in terms of α.

$$\varphi_1 = X_1 + X_2 \sin\alpha \qquad \varphi_2 = X_2 \cos\alpha \longrightarrow \boxed{\varphi(X) = \begin{bmatrix} X_1 + X_2 \sin\alpha \\ X_2 \cos\alpha \end{bmatrix}}$$

2.) Compute the deformation gradient $\underline{\underline{F}}$ and the right Cauchy-Green deformation tensor $\underline{\underline{C}}$

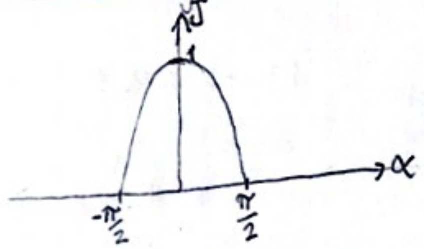
$$\boxed{F = \nabla\varphi = \begin{bmatrix} 1 & \sin\alpha \\ 0 & \cos\alpha \end{bmatrix}}$$

$$C = F^T F = \begin{bmatrix} 1 & \sin\alpha \\ \sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix} = \boxed{\begin{bmatrix} 1 & \sin\alpha \\ \sin\alpha & 1 \end{bmatrix}}$$

3.) Compute and plot the variation in volume of the solid as a function of α.

$$\boxed{J = \det(F) = \cos\alpha} > 0 \longrightarrow \text{variation of volume}$$

$$\underline{\underline{-\frac{\pi}{2} < \alpha < \frac{\pi}{2}}}$$



4.) At what point do the deformations cease to be admissible?

At $\underline{\underline{\alpha = \pm\frac{\pi}{2}}}$ the solid becomes a straight line along the X_1 axis, which implies that it has no volume ($J = 0$) and this is not physically possible.

5.) Compute the change in length of the diagonals AC and BD, and the change in the angle β subtended by them. Interpret geometrically. Plot the change of lengths and the change of angle β as a function of α.

Since the initial shape is $\underline{\text{square}}$ $\longrightarrow$ $AB = BC = CD = DA = L$

Material coordinates of points A, B, C, D $\longrightarrow$ $A=(0,0)$ $B=(L,0)$ $C=(L,L)$ $D=(0,L)$

$\varphi(A)=[0 \quad 0]^T$ $\quad \varphi(B)=[L \quad 0]^T$ $\quad \varphi(C)=[L(1+\sin\alpha) \quad L\cos\alpha]^T$ $\quad \varphi(D)=[L\sin\alpha \quad L\cos\alpha]^T$

From geometrical analysis $\longrightarrow$ $\beta=\frac{\pi}{2}+\tan^{-1}\left(\frac{L-\varphi_1(D)}{\varphi_2(C)}\right)-\tan^{-1}\left(\frac{\varphi_2(C)}{\varphi_1(C)}\right)$

$$\beta=\frac{\pi}{2}+\tan^{-1}\left(\frac{L(1-\sin\alpha)}{L\cos\alpha}\right)-\tan^{-1}\left(\frac{L\cos\alpha}{L(1+\sin\alpha)}\right) \qquad \cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{(1-\sin\alpha)(1+\sin\alpha)}$$

$$\beta=\frac{\pi}{2}+\tan^{-1}\left(\left[\frac{(1-\sin\alpha)^2}{(1-\sin\alpha)(1+\sin\alpha)}\right]^{\frac{1}{2}}\right)-\tan^{-1}\left(\left[\frac{(1-\sin\alpha)(1+\sin\alpha)}{(1+\sin\alpha)^2}\right]^{\frac{1}{2}}\right)\longrightarrow \beta=\frac{\pi}{2}$$

$\boxed{\beta=\frac{\pi}{2} \longrightarrow \text{It remains constant independently from } \alpha}$

$$AC=\sqrt{\varphi_1(C)^2+\varphi_2(C)^2}=\sqrt{L^2(1+\sin\alpha)^2+L^2\cos^2\alpha}=L\sqrt{1+2\sin\alpha+\sin^2\alpha+\cos^2\alpha}$$

$=L\sqrt{2(1+\sin\alpha)}$ $\qquad$ for $\alpha=0 \longrightarrow AC_0=L\sqrt{2}$

$\frac{AC}{AC_0}=\boxed{\sqrt{1+\sin\alpha}} \longrightarrow$ Change of AC (stretch) $\longrightarrow$ Increases as α increases for $0\le\alpha<\frac{\pi}{2}$, AC is subjected to traction stress, for $-\frac{\pi}{2}<\alpha<0$ it is under compression.

$$BD=\sqrt{(L-\varphi_1(D))^2+\varphi_2(D)^2}=\sqrt{(L-L\sin\alpha)^2+L^2\cos^2\alpha}=L\sqrt{(1-2\sin\alpha+\sin^2\alpha)+\cos^2\alpha}$$

$=L\sqrt{2(1-\sin\alpha)}$ $\qquad$ for $\alpha=0 \longrightarrow BD_0=L\sqrt{2}$

$\frac{BD}{BD_0}=\boxed{\sqrt{1-\sin\alpha}} \longrightarrow$ Change of BD (stretch) $\longrightarrow$ Increases as α decreases for $-\frac{\pi}{2}<\alpha\le 0$, BD is subjected to traction, for $0<\alpha<\frac{\pi}{2}$ it is under compression

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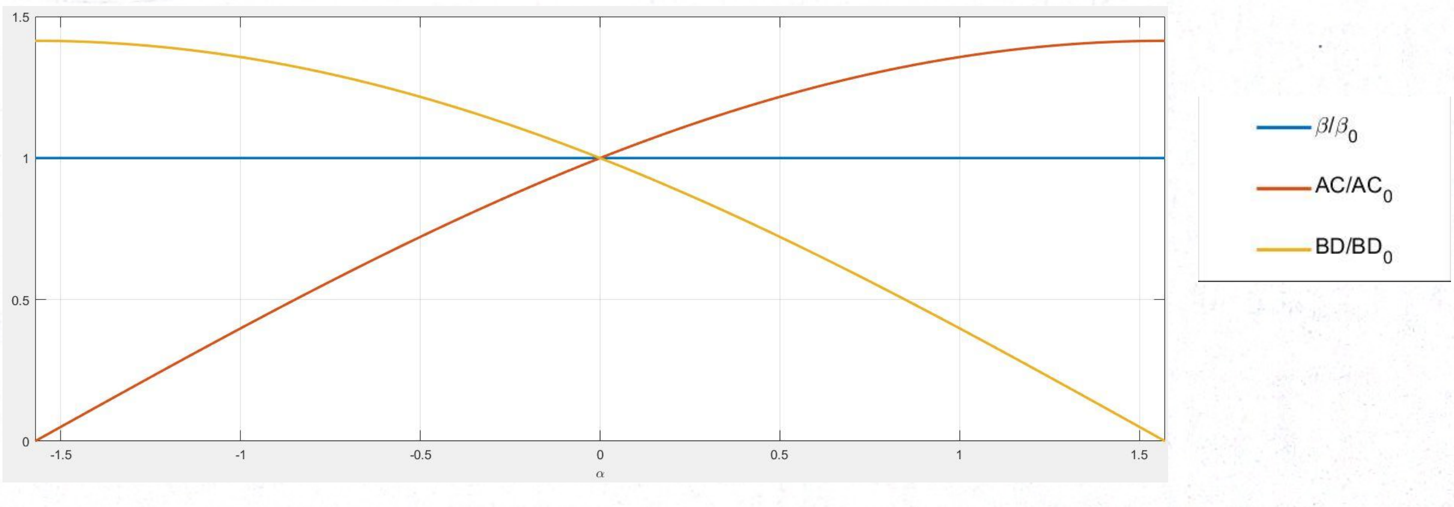

Exercise 3

Consider a cylindrical solid referred to an orthonormal cartesian reference frame $\{X_1, X_2, X_3\}$ whose axis aligned with the X_3 direction. Its normal cross section occupies a region Ω in the $\{X_1, X_2\}$ plane of boundary $\partial\Omega$. An anti-plane shear deformation of the solid can be defined as one for which the deformation mapping is:

$$\varphi_1 = X_1 \qquad \varphi_2 = X_2 \qquad \varphi_3 = X_3 + w(X_1, X_2)$$

1.) Sketch the defamotion of Ω.

a.) Compute the deformation gradient $\underline{\underline{F}}$, the right Cauchy-Green tensor $\underline{\underline{C}}$, and the Jacobian J in terms of w.

$$\underline{\underline{F}} = \nabla \underline{\varphi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 1 \end{bmatrix}$$

$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \begin{bmatrix} 1 + \left(\frac{\partial w}{\partial X_1}\right)^2 & \left(\frac{\partial w}{\partial X_1}\right)\left(\frac{\partial w}{\partial X_2}\right) & \frac{\partial w}{\partial X_1} \\ \left(\frac{\partial w}{\partial X_1}\right)\left(\frac{\partial w}{\partial X_2}\right) & 1 + \left(\frac{\partial w}{\partial X_2}\right)^2 & \frac{\partial w}{\partial X_2} \\ \frac{\partial w}{\partial X_1} & \frac{\partial w}{\partial X_2} & 1 \end{bmatrix}$$

$$J = \det(F) = 1$$

b.) Does the solid change volume during the deformation?

Since the Jacobian is not a function of w $(J = 1)$ the volume does not change.

c.) Are the local impenetrability conditions satisfied?

Since $J > 0$, the impenetrability conditions are satisfied.

2.) Consider the unit vectors $\underline{A} = \frac{w_{,1} \underline{E}_1 + w_{,2} \underline{E}_2}{\sqrt{w_{,1}^2 + w_{,2}^2}}$ $\qquad \underline{B} = \frac{-w_{,2} \underline{E}_1 + w_{,1} \underline{E}_2}{\sqrt{w_{,1}^2 + w_{,2}^2}}$

$$\underline{A} = \frac{\nabla w}{\|\nabla w\|} \qquad \underline{B} = \frac{\left[-\frac{\partial w}{\partial X_2} \ \ \frac{\partial w}{\partial X_1}\right]^T}{\|\nabla w\|}$$

a.) How are $\underline{A}$ and $\underline{B}$ related to the level contours of $w(X_1, X_2)$?

A and B describe the direction of change of w on the $\{X_1, X_2\}$ plane.

b.) Compute (in terms of w) the change in length (measured by the corresponding stretch ratios) of $\underline{A}$ and $\underline{B}$ as well as the change in the angle subtended by $\underline{A}$ and $\underline{B}$

$\underline{A}$ and $\underline{B}$ are unitary vectors and therefore, their length does not change.

$$A \cdot B = \underbrace{\cancel{\|A\|}}_{1} \underbrace{\cancel{\|B\|}}_{1} \cos\alpha \qquad \cos\alpha = A \cdot B = \frac{-w_{,1} w_{,2} + w_{,1} w_{,2}}{\|\nabla w\|} = 0 \qquad \boxed{\alpha = \pm \frac{\pi}{2}}$$

α is always $\pm\frac{\pi}{2}$ and it is not a function of w

3.) Using the Piola transformation, compute the change of area of, and in the normal to, an infinitesimal material area contained in the $\{X_1, X_2\}$ plane

$$ds = J \underline{\underline{F}}^{-T} dS \qquad \underline{\underline{F}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\partial w}{\partial X_1} & -\frac{\partial w}{\partial X_2} & 1 \end{bmatrix} \qquad \underline{\underline{F}}^{-T} = \begin{bmatrix} 1 & 0 & -\frac{\partial w}{\partial X_1} \\ 0 & 1 & -\frac{\partial w}{\partial X_2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$ds\,\underline{n} = (J\,dS)\,\underline{\underline{F}}^{-T} \underline{N}_S \qquad \underline{N} = \underline{E}_3$$

$$ds\,\underline{n} = (J\,dS)\begin{bmatrix} -\frac{\partial w}{\partial X_1} \\ -\frac{\partial w}{\partial X_2} \\ 1 \end{bmatrix} \qquad \boxed{ds\,\underline{n} = dS\left[-\frac{\partial w}{\partial X_1} \;\; -\frac{\partial w}{\partial X_2} \;\; 1\right]^T}$$

$$ds\,\underline{n} = dS\sqrt{w_{,1}^2 + w_{,2}^2 + 1}\;\frac{[-w_{,1} \;\; -w_{,2} \;\; 1]^T}{\sqrt{w_{,1}^2 + w_{,2}^2 + 1}}$$

$$\boxed{\underline{n} = \frac{[-w_{,1} \;\; -w_{,2} \;\; 1]^T}{\sqrt{w_{,1}^2 + w_{,2}^2 + 1}}} \qquad \boxed{ds = \sqrt{w_{,1}^2 + w_{,2}^2 + 1}\; dS}$$

5.) Derive an integral expression for the deformed area of the domain Ω

$$ds = (w_{,1}^2 + w_{,2}^2 + 1)^{1/2}\, d\Omega \qquad ds\text{: differential of deformed area}$$

$$\boxed{\varphi(\Omega) = s = \int (w_{,1}^2 + w_{,2}^2 + 1)^{1/2}\, d\Omega} \qquad d\Omega = dX_1\, dX_2$$

Let the boundary $\partial\Omega$ of Ω be defined parametrically by $X_1 = X_1(S)$, $X_2 = X_2(S)$ where $0 \le S \le L$ is the arc length measured along $\partial\Omega$. Note that $\underline{E}_1 \frac{X_1(S)}{dS} + \underline{E}_2 \frac{X_2(S)}{dS}$ is the unit vector tangent to $\partial\Omega$. Derive an integral expression for the perimeter of the deformed boundary $\varphi(\partial\Omega)$

$$T = \frac{[X_1(S) \;\; X_2(S) \;\; 0]^T}{dS} \qquad \text{stretch} \rightarrow \lambda = \|\lambda_T\| = (T \cdot CT)^{1/2} \qquad ds = \lambda\, dS$$

$$T \cdot CT = \frac{1}{dS^2}\left(X_1^2 + X_2^2 + (X_1 w_{,1} + X_2 w_{,2})^2\right) \qquad \text{(Done in MatLab)}$$

if T is a unitary vector, then $dS = \sqrt{w_1^2 + w_2^2}$ $\longrightarrow T \cdot CT = 1 + \left(\frac{X_1 w_{,1} + X_2 w_{,2}}{dS}\right)^2$

$$\boxed{\lambda = \left[1 + \left(\frac{X_1 w_{,1} + X_2 w_{,2}}{dS}\right)^2\right]^{1/2} \qquad \varphi(\partial\Omega) = \int_0^L \lambda\, dS}$$