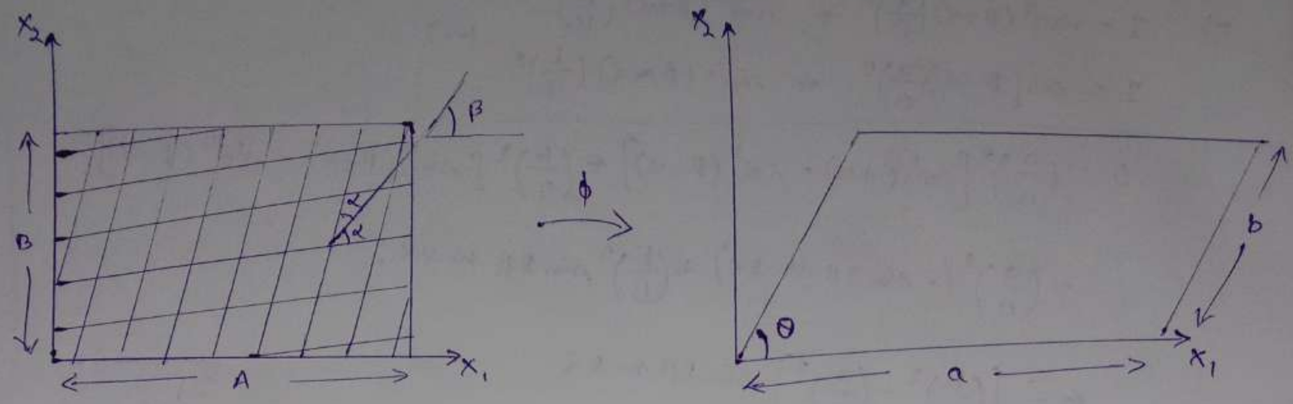
28/04/2019 HW-1 ↔ CSM SHIVAPRASAD MARISWAMY

a)



1) $x_1 = \frac{a}{A} X_1 + \frac{b}{B}\cos\theta \cdot X_2$

$x_2 = X_2 \frac{b \sin\theta}{B}$

deformation mapping. $\bar{x} = \phi(\bar{X}, t) = \begin{bmatrix} \frac{a}{A} X_1 + X_2 \frac{b}{B}\cos\theta \\ X_2 \frac{b}{B}\sin\theta \end{bmatrix}$

$\underline{\underline{F}} = \text{grad}\,\phi = \begin{bmatrix} \frac{a}{A} & \frac{b}{B}\cos\theta \\ 0 & \frac{b}{B}\sin\theta \end{bmatrix}$

Right Cauchy-Green deformation tensor

$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \begin{bmatrix} \frac{a}{A} & 0 \\ \frac{b}{B}\cos\theta & \frac{b}{B}\sin\theta \end{bmatrix} \begin{bmatrix} \frac{a}{A} & \frac{b}{B}\cos\theta \\ 0 & \frac{b}{B}\sin\theta \end{bmatrix} = \begin{bmatrix} \left(\frac{a}{A}\right)^2 & \frac{ab}{AB}\cos\theta \\ \frac{ab}{AB}\cos\theta & \left(\frac{b}{B}\right)^2 \end{bmatrix}$ //.

2) As the fibres are inextensible

$\lambda^2 = 1 = \underline{N} \cdot \underline{\underline{C}} \cdot \underline{N} = N_I \cdot C_{IJ} \cdot N_J.$

where, $N_I = \begin{pmatrix} \cos(\beta+\alpha) \\ \sin(\beta+\alpha) \end{pmatrix}$, $N_J = \begin{pmatrix} \cos(\beta-\alpha) \\ \sin(\beta-\alpha) \end{pmatrix}$

For $\theta = 90^\circ$, $\cos\theta = 0$.

$\therefore \underline{\underline{C}} = \begin{bmatrix} \left(\frac{a}{A}\right)^2 & 0 \\ 0 & \left(\frac{b}{B}\right)^2 \end{bmatrix}$ $\Rightarrow \lambda^2 = \begin{bmatrix} \cos(\beta+\alpha) \\ \sin(\beta+\alpha) \end{bmatrix}^T \begin{bmatrix} \left(\frac{a}{A}\right)^2 & 0 \\ 0 & \left(\frac{b}{B}\right)^2 \end{bmatrix} \begin{bmatrix} \cos(\beta+\alpha) \\ \sin(\beta+\alpha) \end{bmatrix}$

$\therefore$ $\cancel{\lambda^2 = \cos(\beta-\alpha)\left(\frac{a}{A}\right)^2 \cos(\beta+\alpha) + \left(\frac{b}{B}\right)^2 \sin(\beta-\alpha)\sin(\beta+\alpha) = 1}$

$\Rightarrow \left(\frac{a}{A}\right)^2 (\cos\beta\cos\alpha - \sin\beta\sin\alpha)(\cos\beta\cos\alpha + \sin\beta\sin\alpha) + \left(\frac{b}{B}\right)^2 (\sin\beta\cos\alpha - \cos\beta\sin\alpha)(\sin\beta\cos\alpha + \cos\beta\sin\alpha) = 1$

$$\begin{bmatrix}\cos(\beta-\alpha)\\ \sin(\beta-\alpha)\end{bmatrix}^T \begin{bmatrix}\left(\frac{a}{A}\right)^2 & 0\\ 0 & \left(\frac{b}{B}\right)^2\end{bmatrix}\begin{bmatrix}\cos(\beta-\alpha)\\ \sin(\beta-\alpha)\end{bmatrix}$$

$$\Rightarrow \quad I = \cos^2(\beta+\alpha)\left(\frac{a}{A}\right)^2 + \sin^2(\beta+\alpha)\left(\frac{b}{B}\right)^2 \qquad (-)$$

$$I = \cos(\beta-\alpha)\left(\frac{a}{A}\right)^2 + \sin^2(\beta-\alpha)\left(\frac{b}{B}\right)^2$$

$$0 = \left(\frac{a}{A}\right)^2\left[\cos^2(\beta+\alpha) - \cos^2(\beta-\alpha)\right] + \left(\frac{b}{B}\right)^2\left[\sin^2(\beta+\alpha) - \sin^2(\beta-\alpha)\right].$$

$$= \left(\frac{a}{A}\right)^2(-\sin 2\beta \sin 2\alpha) + \left(\frac{b}{B}\right)^2 \sin 2\beta \sin 2\alpha.$$

$$0 = \left[\left(\frac{b}{B}\right)^2 - \left(\frac{a}{A}\right)^2\right]\sin 2\beta \sin 2\alpha$$

$$\Rightarrow \sin 2\beta = 0$$

$$\therefore \quad \beta = \frac{n\pi}{2} \qquad \forall\, n = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

3) $\lambda^2 = 1 = NCN$

$$\Rightarrow \quad 1 = \left(\frac{a}{A}\right)^2\cos^2(\beta+\alpha) + \frac{2ab}{AB}\sin(\beta+\alpha)\cos(\beta+\alpha)\cos\theta + \left(\frac{b}{B}\right)^2\sin^2(\beta+\alpha) \quad — ①$$

$$1 = \left(\frac{a}{A}\right)^2\cos^2(\beta-\alpha) + \frac{2ab}{AB}\cos(\beta-\alpha)\sin(\beta-\alpha)\cos\theta + \left(\frac{b}{B}\right)^2\sin^2(\beta-\alpha) \quad — ②$$

Solving eqns ① and ②

$$\left[\left(\frac{b}{B}\right)^2 - \left(\frac{a}{A}\right)^2\right]\sin 2\beta \sin 2\alpha + \frac{2ab}{AB}\sin 2\alpha \cos 2\beta \cos\theta = 0$$

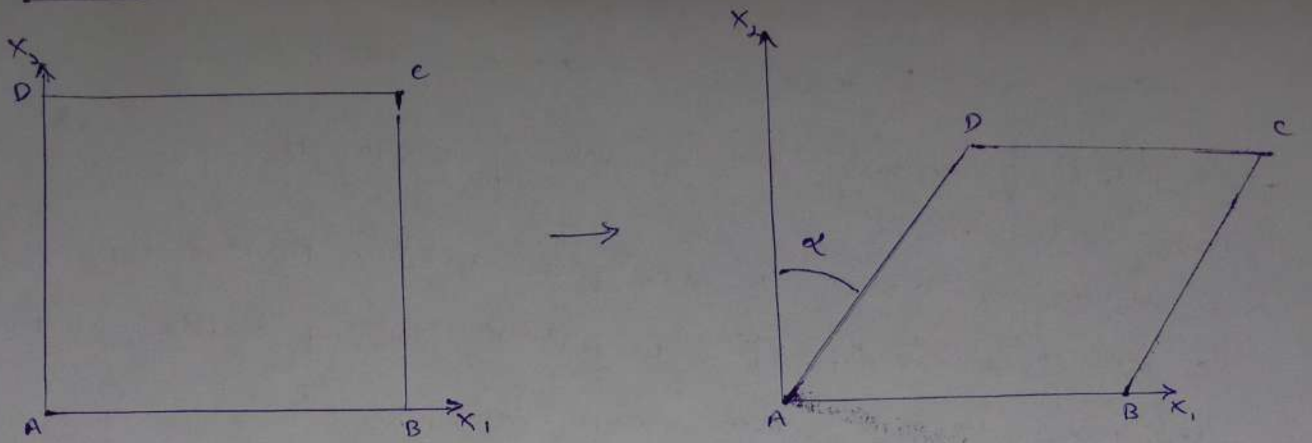
4) $$\underline{\underline{C}} = \begin{bmatrix}\left(\frac{a}{A}\right)^2 & \frac{ab}{AB}\cos\theta\\ \frac{ab}{AB}\cos\theta & \left(\frac{b}{B}\right)^2\end{bmatrix}$$

$$E = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) = \frac{1}{2}\begin{bmatrix}\left(\frac{a}{A}\right)^2 - 1 & \frac{ab}{AB}\cos\theta\\ \frac{ab}{AB}\cos\theta & \left(\frac{b}{B}\right)^2 - 1\end{bmatrix}$$

$$\therefore \quad \nu = -\frac{E_{22}}{E_{11}} \quad \cancel{= -\frac{\left(\frac{a}{A}\right)^2 - 1}{\left(\frac{a}{A}\right)^2 - 1}} \quad = \frac{-\left(\left(\frac{b}{B}\right)^2 - 1\right)}{\left(\frac{a}{A}\right)^2 - 1} = \frac{\frac{B^2 - b^2}{B^2}}{\frac{a^2 - A^2}{A^2}}$$

$$\nu = \frac{A^2(B^2 - b^2)}{B^2(a^2 - A^2)} \quad //$$

b) 2-D solid



1.) From the geometry,

$$x_1 = X_1 + X_2 \sin\alpha$$

$$x_2 = X_2 \cos\alpha$$

Hence, the deformation mapping can be written as

$$\bar{x} = \phi(\bar{X}, t) = \begin{bmatrix} X_1 + X_2 \sin\alpha \\ X_2 \cos\alpha \end{bmatrix}$$

2) $\underline{\underline{F}} = \text{grad}\,\phi$

$$\underline{\underline{F}} = \begin{bmatrix} \frac{\partial \phi_x}{\partial X} & \frac{\partial \phi_x}{\partial Y} \\ \frac{\partial \phi_y}{\partial X} & \frac{\partial \phi_y}{\partial Y} \end{bmatrix} = \begin{bmatrix} 1 & \sin\alpha \\ 0 & \cos\alpha \end{bmatrix}$$

$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \begin{bmatrix} 1 & 0 \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & \sin\alpha \\ 0 & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & \sin\alpha \\ \sin\alpha & 1 \end{bmatrix}$$ is the Right Cauchy-Green deformation tensor.

3) For the given 2-D solid

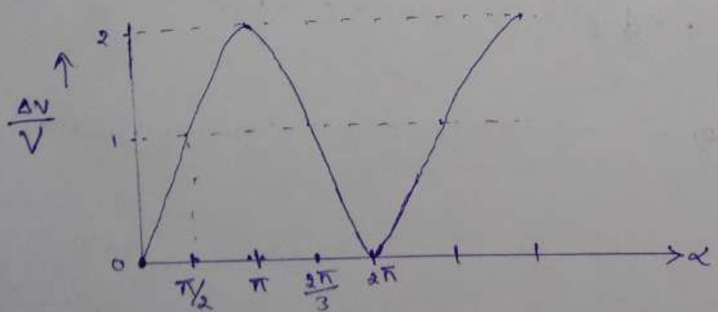
$$dv = J\,dV$$

$$\Rightarrow v = JV = (\det F)\,V$$

$$\therefore \quad v = V\cos\alpha.$$

variation in volume $= \cancel{V - V(\cos\alpha - 1)}. = \cancel{V(1-\cos\alpha)}$

$$\Delta V = V - v = V(1 - \cos\alpha)$$



4) Deformations are admissible for $J>0$, (reality)

Hence, it ceases to be admissible when $J<0$

Nothing but, $J = \cos\alpha < 0$ when $\alpha > 90°$

→ To interpret geometrically, for any value of $\alpha < 90°$ the square solid will have positive volume.

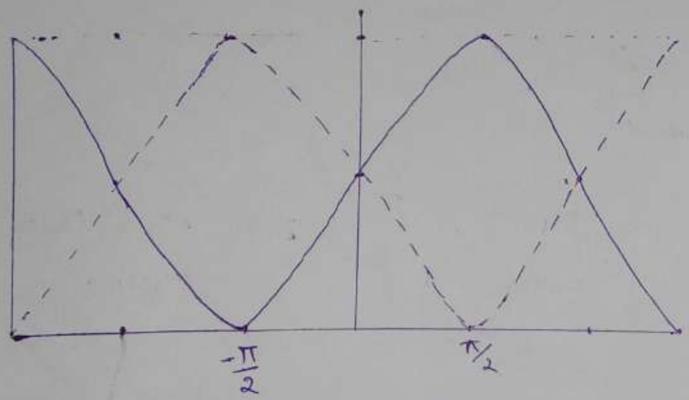
→ At the moment $\alpha = 90°$ the sides of the 2D solid DC and AB fall on the coordinate X_1. This is just the shape of a line, 1-D geometry. 1-D geometry cannot contain any volume.

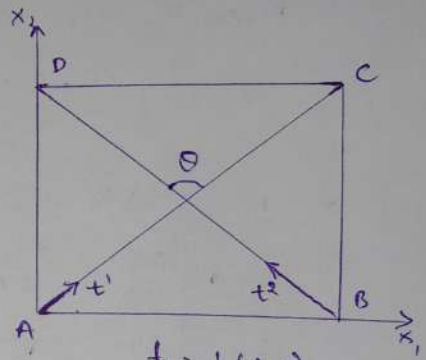
→ For values of $\alpha > 90°$, side DC has to cross its opposite side AB. This kind of twisting is physically not possible.

5) stretch, $\lambda = \sqrt{t \cdot C t}$

$\Rightarrow$ $\lambda_{AC} = 1 + \sin\alpha$ along AC

$\lambda_{BD} = 1 - \sin\alpha$ along BD





$t_1 = \frac{1}{\sqrt{2}}(1 \;\; 1)$

$t_2 = \frac{1}{\sqrt{2}}(-1 \;\; 1)$

for angle θ subtended b/w the two diagonals,

$$\cos\theta = \frac{t^{(1)} \cdot (I + 2E)\, t^{(2)}}{\sqrt{1 + 2t^{(1)} \cdot E t^{(1)}}\ \sqrt{1 + 2t^{(2)} \cdot E t^{(2)}}}$$

and $E = \frac{1}{2}(\underline{C} - \underline{I}) = \frac{1}{2}\begin{bmatrix} 0 & \sin\alpha \\ \sin\alpha & 0 \end{bmatrix}$

$\sqrt{1 + 2t^{(1)} \cdot E t^{(1)}} = \sqrt{1 + \sin\alpha}$

$\sqrt{1 + 2t^{(2)} \cdot E t^{(2)}} = \sqrt{1 - \sin\alpha}$

$t^{(1)} \cdot (I + 2E)\, t^{(2)} = \frac{1}{2}(1, 1)\begin{bmatrix} \sin\alpha - 1 \\ 1 - \sin\alpha \end{bmatrix} = 0 \;\forall\, \alpha. \Rightarrow \cos\theta = 0 \;\; \forall \alpha$

This implies that the angle θ does not change and remains a right angle.