Computational Structural Mechanics and Dynamics

Homework I

Student's Name

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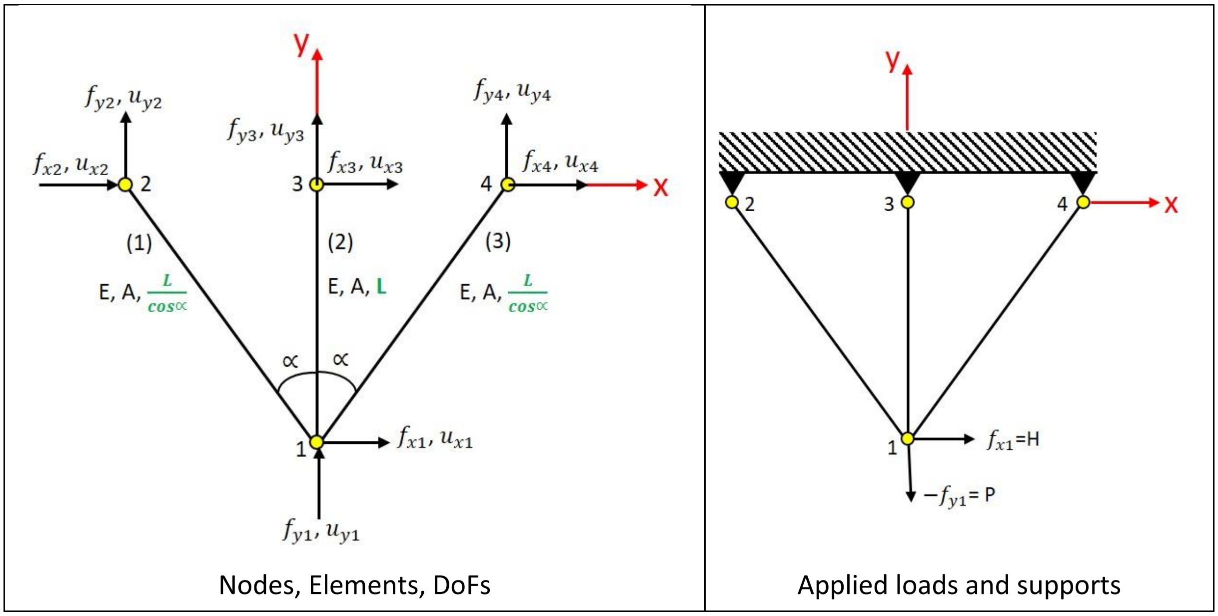
Assignment 1

The finite element model for the problem is shown in the following:



As it is obvious from the figures, the structure has 4 nodes each of which has 2 degrees of freedom meaning that the whole structure has 8 DoFs. Therefore, the global stiffness matrix is 8 by 8 and the global equation can be written as the following:

$$f = \left[f_{x1}\ f_{y1}\ f_{x2}\ f_{y2}\ f_{x3}\ f_{y3}\ f_{x4}\ f_{y4}\right]^T$$

$$u = \left[u_{x1}\ u_{y1}\ u_{x2}\ u_{y2}\ u_{x3}\ u_{y3}\ u_{x4}\ u_{y4}\right]^T$$

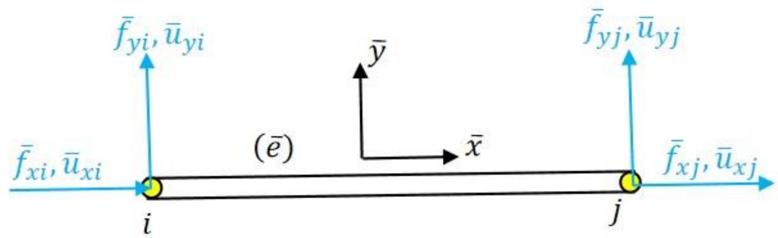
$$K = \begin{bmatrix} K_{x1x1} & K_{x1y1} & K_{x1x2} & K_{x1y2} & K_{x1x3} & K_{x1y3} & K_{x1x4} & K_{x1y4} \\ K_{y1x1} & K_{y1y1} & K_{y1x2} & K_{y1y2} & K_{y1x3} & K_{y1y3} & K_{y1x4} & K_{y1y4} \\ K_{x2x1} & K_{x2y1} & K_{x2x2} & K_{x2y2} & K_{x2x3} & K_{x2y3} & K_{x2x4} & K_{x2y4} \\ K_{y2x1} & K_{y2y1} & K_{y2x2} & K_{y2y2} & K_{y2x3} & K_{y2y3} & K_{y2x4} & K_{y2y4} \\ K_{x3x1} & K_{x3y1} & K_{x3x2} & K_{x3y2} & K_{x3x3} & K_{x3y3} & K_{x3x4} & K_{x3y4} \\ K_{y3x1} & K_{y3y1} & K_{y3x2} & K_{y3y2} & K_{y3x3} & K_{y3y3} & K_{y3x4} & K_{y3y4} \\ K_{x4x1} & K_{x4y1} & K_{x4x2} & K_{x4y2} & K_{x4x3} & K_{x4y3} & K_{x4x4} & K_{x4y4} \\ K_{y4x1} & K_{y4y1} & K_{y4x2} & K_{y4y2} & K_{y4x3} & K_{y4y3} & K_{y4x4} & K_{y4y4} \end{bmatrix}$$

$$f_{8\times1} = K_{8\times8} \times u_{8\times1}$$

with f and u defined as nodal forces and nodal displacements, respectively.

In order to form the global equation, we need to find the elemental stiffness matrices. The member stiffness equation for an element in arbitrary angle φ, is obtained using the following procedure:

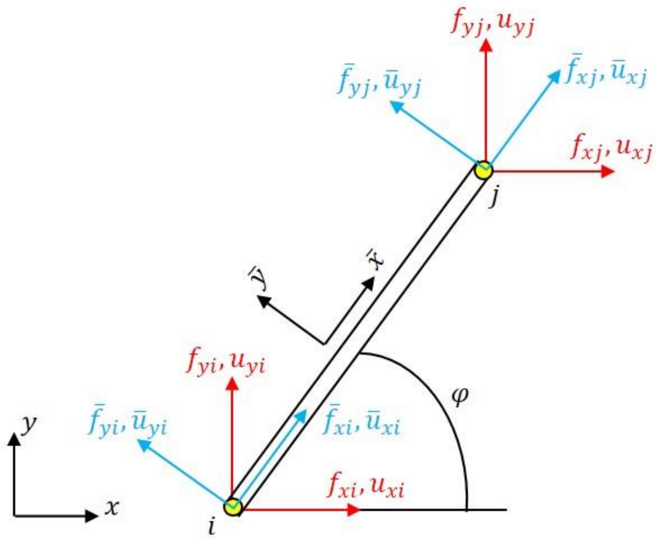
Consider the bar of length L, area A and Young modulus E as shown in the figure. For this bar the elemental stiffness matrix can be written according to the nodal forces and displacements:



$$\begin{bmatrix} \bar{f}_{x1} \\ \bar{f}_{y1} \\ \bar{f}_{x2} \\ \bar{f}_{y2} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{u}_{x2} \\ \bar{u}_{y2} \end{bmatrix}$$

while the bar has two nodes and each node has two DoFs, the stiffness matrix is 4 by 4.

Now, consider a 2-noded element with arbitrary angle φ with respect to x axis. The nodal displacements and forces both need transformation to standard axis. These transformations are done using the following transformation matrix



$$T^{(e)} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

where $c = cos(\varphi)$ and $s = sin(\varphi)$ such that

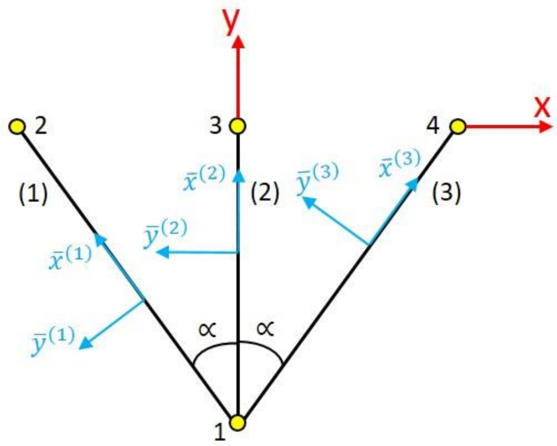
$$\bar{u}^{(e)} = T^{(e)} u^{(e)}$$

$$f^{(e)} = \left(T^{(e)}\right)^T \bar{f}^{(e)}$$

Therefore, the transformed elemental stiffness matrix for any bar of angle φ is

$$\left(T^{(e)}\right)^T \bar{K}^{(e)} T^{(e)} = \frac{E^{(e)} A^{(e)}}{L^{(e)}} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

In our problem, we have three bars whose angles, based on the figure, are written as the following



Element Number	Angle (φ)	Trigonometric Relations and Element Length
(1)	$\varphi = \frac{\pi}{2} + \alpha$	$\cos(\varphi) = -\sin(\alpha)\,;\,\sin(\varphi) = cos(\alpha)$ $L^{(e)} = \frac{L}{\cos(\alpha)}$
(2)	$\varphi = \frac{\pi}{2}$	$\cos(\varphi) = 0\,;\,\sin(\varphi) = 1$ $L^{(e)} = L$
(3)	$\varphi = \frac{\pi}{2} - \alpha$	$\cos(\varphi) = \sin(\alpha)\,;\,\sin(\varphi) = cos(\alpha)$ $L^{(e)} = \frac{L}{\cos(\alpha)}$

Therefore, taking into account that the bars have the same A and E, the element stiffness matrix for each bar is written in terms of $\sin(\alpha)$ and $\cos(\alpha)$ as:

$$K^{(1)} = \frac{EA}{L}\begin{bmatrix} cs^2 & -sc^2 & -cs^2 & sc^2 \\ -sc^2 & c^3 & sc^2 & -c^3 \\ -cs^2 & sc^2 & cs^2 & -sc^2 \\ sc^2 & -c^3 & -sc^2 & c^3 \end{bmatrix}$$

$$K^{(2)} = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$K^{(3)} = \frac{EA}{L}\begin{bmatrix} cs^2 & sc^2 & -cs^2 & -sc^2 \\ sc^2 & c^3 & -sc^2 & -c^3 \\ -cs^2 & -sc^2 & cs^2 & sc^2 \\ -sc^2 & -c^3 & sc^2 & c^3 \end{bmatrix}$$

In order to proceed with the assembly process, we should write the expanded element stiffness equations and then reconnect members by compatibility rule. The expanded element stiffness equations are

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{x3}^{(1)} \\ f_{y3}^{(1)} \\ f_{x4}^{(1)} \\ f_{y4}^{(1)} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} cs^2 & -sc^2 & -cs^2 & sc^2 & 0 & 0 & 0 & 0 \\ -sc^2 & c^3 & sc^2 & -c^3 & 0 & 0 & 0 & 0 \\ -cs^2 & sc^2 & cs^2 & -sc^2 & 0 & 0 & 0 & 0 \\ sc^2 & -c^3 & -sc^2 & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

$$\begin{bmatrix} f_{x1}^{(2)} \\ f_{y1}^{(2)} \\ f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \\ f_{x4}^{(2)} \\ f_{y4}^{(2)} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x2}^{(3)} \\ f_{y2}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \\ f_{x4}^{(3)} \\ f_{y4}^{(3)} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} cs^2 & sc^2 & 0 & 0 & 0 & 0 & -cs^2 & sc^2 \\ sc^2 & c^3 & 0 & 0 & 0 & 0 & -sc^2 & -c^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -cs^2 & -sc^2 & 0 & 0 & 0 & 0 & cs^2 & sc^2 \\ -sc^2 & -c^3 & 0 & 0 & 0 & 0 & sc^2 & c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

From equilibrium rule, taking into account the expanded element stiffness equations, we have:

$$f = f^{(1)} + f^{(2)} + f^{(3)} = \left(K^{(1)} + K^{(2)} + K^{(3)}\right)u = Ku$$

Therefore the master stiffness equation is:

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & sc^2 & 0 & 0 & -cs^2 & sc^2 \\ & 1+2c^3 & sc^2 & -c^3 & 0 & -1 & -sc^2 & -c^3 \\ & & cs^2 & -sc^2 & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & symm. & & & 1 & 0 & 0 \\ & & & & & & cs^2 & sc^2 \\ & & & & & & & c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

The 5th row and column corresponds the horizontal displacement of node 3. The horizontal forces are taken by nodes 1 and 4 which means node 3 doesn't tolerate any horizontal reaction therefore it can't move in horizontal direction.

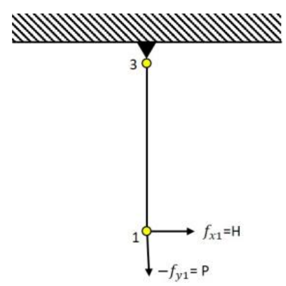
The boundary conditions and applied loads for this problem are:

Displacement BC	$u_{x2} = u_{y2} = u_{x3} = u_{y3} = u_{x4} = u_{y4} = 0$
Force BC	$f_{x1} = H$ $f_{y1} = -P$

We strike out the rows and columns which pertain to known displacement. Therefore, the reduced stiffness equation is

$$\begin{bmatrix} H \\ -P \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} \Rightarrow \begin{cases} u_{x1} = \dfrac{HL}{2cs^2EA} \\ u_{y1} = -\dfrac{PL}{(1+2c^3)EA} \end{cases}$$

If $H \neq 0$ and $\alpha \to 0 \Rightarrow u_{x1} \to \infty$ and $u_{y1} = -\frac{PL}{3EA}$. This means that the solution blows up which is because of the fact that a pin cannot tolerate bending moment but as we have horizontal force a moment appears at the pin. And also the vertical displacement is then divided between the three bars which are now in contact due to $\alpha = 0$.



If $\alpha \to \frac{\pi}{2} \Rightarrow u_{x1} \to \infty$ and $u_{y1} = -\frac{PL}{EA}$.

The axial forces are then found as posteriori process. In order to do this we have to do the following steps:

1st element:

1) $\bar{u}^{(1)} = T^{(1)}u^{(1)} \Rightarrow \bar{u}^{(1)} = \begin{bmatrix} -s & c & 0 & 0 \\ -c & -s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{bmatrix} \begin{bmatrix} \frac{HL}{2cs^2EA} \\ -\frac{PL}{(1+2c^3)EA} \\ 0 \\ 0 \end{bmatrix} = \frac{L}{EA} \begin{bmatrix} \frac{-H}{2cs} - \frac{Pc}{1+2c^3} \\ \frac{-H}{2cs} + \frac{Ps}{1+2c^3} \\ 0 \\ 0 \end{bmatrix}$

2) Elongation: $d^{(1)} = \bar{u}_{x2}{}^{(1)} - \bar{u}_{x1}{}^{(1)} = 0 - \left(\frac{L}{EA}\left(\frac{-H}{2cs} - \frac{Pc}{1+2c^3}\right)\right)$

3) Axial force: $F^{(1)} = \frac{EA}{L/c} d^{(1)} = \frac{H}{2s} + \frac{Pc^2}{1+2c^3}$

2nd element:

1) $\bar{u}^{(2)} = T^{(2)}u^{(2)} \Rightarrow \bar{u}^{(2)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{HL}{2cs^2EA} \\ -\frac{PL}{(1+2c^3)EA} \\ 0 \\ 0 \end{bmatrix} = \frac{L}{EA} \begin{bmatrix} -\frac{P}{1+2c^3} \\ \frac{-H}{2cs^2} \\ 0 \\ 0 \end{bmatrix}$

2) Elongation: $d^{(2)} = \bar{u}_{x2}{}^{(2)} - \bar{u}_{x1}{}^{(2)} = 0 - \left(\frac{L}{EA}\left(0 - \frac{P}{1+2c^3}\right)\right)$

3) Axial force: $F^{(2)} = \frac{EA}{L} d^{(2)} = \frac{P}{1+2c^3}$

3rd element:

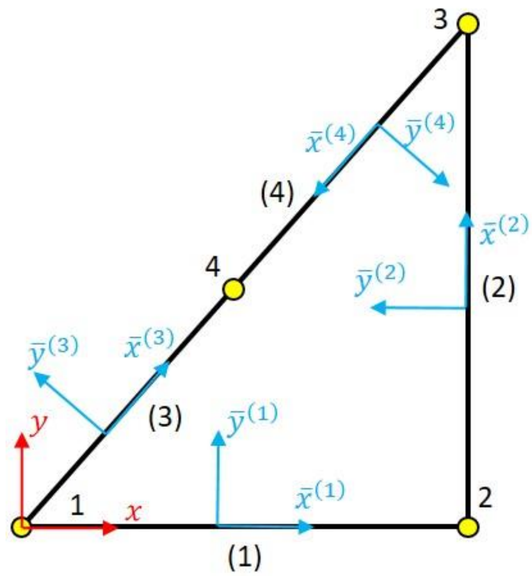
1) $\bar{u}^{(3)} = T^{(3)}u^{(3)} \Rightarrow \bar{u}^{(3)} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} \frac{HL}{2cs^2EA} \\ -\frac{PL}{(1+2c^3)EA} \\ 0 \\ 0 \end{bmatrix} = \frac{L}{EA} \begin{bmatrix} \frac{H}{2cs} - \frac{Pc}{1+2c^3} \\ \frac{-H}{2cs} - \frac{Ps}{1+2c^3} \\ 0 \\ 0 \end{bmatrix}$

2) Elongation: $d^{(3)} = \bar{u}_{x2}{}^{(3)} - \bar{u}_{x1}{}^{(3)} = 0 - \left(\frac{L}{EA}\left(\frac{H}{2cs} - \frac{Pc}{1+2c^3}\right)\right)$

3) Axial force: $F^{(3)} = \frac{EA}{L/c} d^{(3)} = \frac{-H}{2s} + \frac{Pc^2}{1+2c^3}$

If $H \neq 0$ as $\propto \rightarrow 0$ we can see from the solution of $F^{(1)}$ and $F^{(3)}$ that $\sin(\propto) \rightarrow 0$ which is the denominator of one term in $F^{(1)}$ and $F^{(3)}$ which means $F^{(1)}$and $F^{(3)} \rightarrow \infty$. So the solution blows up which is because as the 3 bars are in the same position the whole structure is not restricted in x direction however a force of value H is applied to it.

Assignment 2



Element Number	Angle (φ)	Trigonometric Relations and Element Length
(1)	$\varphi = 0$	$\mathrm{EA} = 100$ $L^{(e)} = 10$
(2)	$\varphi = \frac{\pi}{2}$	$\mathrm{EA} = 50$ $L^{(e)} = 10$
(3)	$\varphi = \alpha$	$\mathrm{EA} = 200\sqrt{2}$ $L^{(e)} = 5\sqrt{2}$
(4)	$\varphi = \pi + \alpha$	$\mathrm{EA} = 200\sqrt{2}$ $L^{(e)} = 5\sqrt{2}$

The globalized element stiffness matrix can be written as the following according to the truss example:

$$K^{(1)} = \begin{bmatrix} 10 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 5 \end{bmatrix}$$

$$K^{(3)} = \begin{bmatrix} 20 & 20 & -20 & -20 \\ 20 & 20 & -20 & -20 \\ -20 & -20 & 20 & 20 \\ -20 & -20 & 20 & 20 \end{bmatrix}$$

$$K^{(4)} = \begin{bmatrix} 20 & 20 & -20 & -20 \\ 20 & 20 & -20 & -20 \\ -20 & -20 & 20 & 20 \\ -20 & -20 & 20 & 20 \end{bmatrix}$$

The expanded element stiffness equations are written as:

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{x3}^{(1)} \\ f_{y3}^{(1)} \\ f_{x4}^{(1)} \\ f_{y4}^{(1)} \end{bmatrix} = \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

$$\begin{bmatrix} f_{x1}^{(2)} \\ f_{y1}^{(2)} \\ f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \\ f_{x4}^{(2)} \\ f_{y4}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x2}^{(3)} \\ f_{y2}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \\ f_{x4}^{(3)} \\ f_{y4}^{(3)} \end{bmatrix} = \begin{bmatrix} 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -20 & -20 & 0 & 0 & 0 & 0 & 20 & 20 \\ -20 & -20 & 0 & 0 & 0 & 0 & 20 & 20 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

$$
\begin{bmatrix} f_{x1}^{(4)} \\ f_{y1}^{(4)} \\ f_{x2}^{(4)} \\ f_{y2}^{(4)} \\ f_{x3}^{(4)} \\ f_{y3}^{(4)} \\ f_{x4}^{(4)} \\ f_{y4}^{(4)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & 0 & -20 & -20 & 20 & 20 \\ 0 & 0 & 0 & 0 & -20 & -20 & 20 & 20 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}
$$

The master stiffness equation based on equilibrium is found as:

$$
\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{bmatrix} = \begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ -10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & -5 & 20 & 25 & -20 & -20 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}
$$

The boundary conditions and applied loads are:

Displacement BC	$u_{x1} = u_{y1} = u_{y2} = 0$
Force BC	$f_{x2} = 0$ $f_{x3} = 2$ $f_{y3} = 1$

Applying these condition to the master equation and reducing the system based on the boundary conditions on displacement yield:

$$
\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 20 & 20 & -20 & -20 \\ 0 & 20 & 25 & -20 & -20 \\ 0 & -20 & -20 & 40 & 40 \\ 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}
$$

The determinant of the reduced stiffness matrix is zero (or the last rows of reduced stiffness matrix are dependent) therefore it is singular. Adding a node in the middle of the bar makes the structure instable.