Computational Structural Mechanics and Dynamics

Assignment 1

The Direct Stiffness Method I

By

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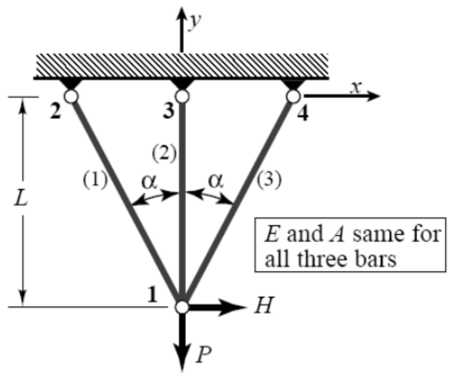
Master in numerical method in engineering

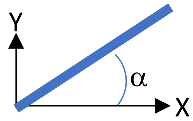
Assignment 1:

Consider the truss problem defined in the Figure. All geometric and material properties: *L, α, E* and *A*, as well as the applied forces *P* and *H*, are to be kept as variables. This truss has 8 degrees of freedom, with six of them removable by the fixed-displacement conditions at nodes 2, 3 and 4. This structure is statically indeterminate as long as $\alpha \neq 0$.



Elemental stiffness matrix

According to the slides of Professor Cervera, the elemental stiffness matrix of an oblique bar is:



$$\mathbf{K}^e = \frac{E^e A^e}{L^e}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

Where:

c = cos (α)

s = sin (α)

Bar 1

In order to define the elemental stiffness matrix in terms of c and s, it will be necessary to use the trigonometric identity written below.

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$
$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

The angle of bar 1 respect to the global coordinate system is 90+α. Using the identity above it is obtained the next relationship.

$$\cos(90+\alpha) = \cos(90)\cos(\alpha) - \sin(90)\sin(\alpha) = -\sin(\alpha)$$
$$\sin(90+\alpha) = \sin(90)\cos(\alpha) + \cos(90)\sin(\alpha) = \cos(\alpha)$$

Taking into account the elemental stiffness matrix of an oblique bar and the relation before, then it is obtained the elemental stiffness matrix for bar 1:

$$K_1 = \frac{EA}{L}c\begin{bmatrix} s^2 & -cs & -s^2 & cs & 0 & 0 & 0 & 0 \\ -cs & c^2 & cs & -c^2 & 0 & 0 & 0 & 0 \\ -s^2 & cs & s^2 & -cs & 0 & 0 & 0 & 0 \\ cs & -c^2 & -cs & c^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Remark 1: It was named c = - sin (α) =-s and s= cos (α) =c.

Remark 2: In order to simplify the assembly operation, it increased the elemental stiffness matrix.

Remark 3: $L^e = L/\cos(\alpha)$ or $L^e = L/c$.

Bar 2

The Bar 2 is a vertical bar, so, the elemental stiffness matrix is:

$$K_2 = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Remark 1: As it has seen in the stiffness matrix of bar 1, it increased the elemental stiffness matrix.

Bar 3

Since bar 3 is an oblique bar, it will be used the same concept used for the bar 1.

The angle of bar 3 respect to the global coordinate system is 90-α. Using the identity above it is obtained the next relationship.

$$\cos(90-\alpha) = \cos(90)\cos(\alpha) + \sin(90)\sin(\alpha) = \sin(\alpha)$$
$$\sin(90-\alpha) = \sin(90)\cos(\alpha) - \cos(90)\sin(\alpha) = \cos(\alpha)$$

Taking into account the elemental stiffness matrix of an oblique bar and the relation before, then it is obtained the elemental stiffness matrix for bar 3:

$$K_3 = \frac{EA}{L}c\begin{bmatrix} s^2 & cs & 0 & 0 & 0 & 0 & -s^2 & -cs \\ cs & c^2 & 0 & 0 & 0 & 0 & -cs & -c^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -s^2 & -cs & 0 & 0 & 0 & 0 & s^2 & cs \\ -cs & -c^2 & 0 & 0 & 0 & 0 & cs & c^2 \end{bmatrix}$$

Remark 1: It was named c = sin (α) = s and s= cos (α) =c.

Remark 2: In order to simplify the assembly operation, it increased the elemental stiffness matrix.

Remark 3: $L^e = L/\cos(\alpha)$ or $L^e = L/c$.

Assembly process

a) As the elemental stiffness matrices have been prepared for simplifying this process, then it is obtained the global stiffness matrix as $K_{glob} = K_1 + K_2 + K_3$

$$K_{glob} = \frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ 0 & 2c^3+1 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ -cs^2 & c^2s & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ c^2s & -c^3 & -c^2s & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -cs^2 & -c^2s & 0 & 0 & 0 & 0 & cs^2 & c^2s \\ -c^2s & -c^3 & 0 & 0 & 0 & 0 & c^2s & c^3 \end{bmatrix}$$

As it can be seen the 5th row and column contain only zeros, the physical meaning is corresponded to the fact that there is no horizontal reaction force on the node 2, so, there is no contribution in the stiffness of the system. Only the horizontal reaction forces of the node 1 and 3 counterbalance the effect of the horizontal force called H in order to satisfied the equilibrium equation in the x direction.

b) The BC of the problem are:

Node 2 $u_{x2} = u_{y2} = 0$

Node 3 $u_{x3} = u_{y3} = 0$

Node 4 $u_{x4} = u_{y4} = 0$

Applying the BC, the system of equation will be reduced to:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{x2} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

Where:

$$\hat{K}=\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix} \quad \hat{u}=\begin{bmatrix} u_{x1} \\ u_{x2} \end{bmatrix} \quad \hat{f}=\begin{bmatrix} H \\ -P \end{bmatrix}$$

c) Solving the system of equation written above, the unknown displacements will be:

$$u_{x1}=\frac{H}{2cs^2}\frac{L}{EA},\ u_{y1}=\frac{-P}{(1+2c^3)}\frac{L}{EA}$$

Limits cases α = 0 and α = $\pi/2$:

1) α = 0

$$\lim_{\alpha\to 0} u_{x1}=\lim_{\alpha\to 0}\frac{H}{2cs^2}\frac{L}{EA}\to\infty$$

$$\lim_{\alpha\to 0} u_{y1}=\lim_{\alpha\to 0}\frac{-P}{(1+2c^3)}\frac{L}{EA}=\frac{-PL}{3EA}$$

2) α = $\pi/2$

$$\lim_{\alpha\to\frac{\pi}{2}} u_{x1}=\lim_{\alpha\to\frac{\pi}{2}}\frac{H}{2cs^2}\frac{L}{EA}\to\infty$$

$$\lim_{\alpha\to\frac{\pi}{2}} u_{y1}=\lim_{\alpha\to\frac{\pi}{2}}\frac{-P}{(1+2c^3)}\frac{L}{EA}=\frac{-PL}{EA}$$

In the case of α = 0, u_{x1} tend to infinite due to the structure becomes in a mechanism where the nodes 2, 3 and 4 concur at the same node, in this case at node 3. On the other hand, the displacement u_{y1} is 3 time less than the displacement obtained in the case 2. The reason is due to the stiffness of the system (α = 0) is 3 time larger than the system obtained with α = $\pi/2$.

d) Axial Forces in the three members.

For each member, it will necessary extract u^e from u and transform them to local displacement taking into account the expression written below:

$$\bar{u}^e=T^e u^e$$

Where:

$$T^e=\begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

Once obtained $\bar{u}^e$, then, compute d as $d=\bar{u}_{xj}^{\,e}-\bar{u}_{xi}^{\,e}$ and finally compute the axial force as $F=d\frac{E^eA^e}{L^e}$

Axial Force in Bar 1

$$T_1 = \begin{bmatrix} -s & c & 0 & 0 \\ -c & -s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{bmatrix} \quad u_1 = \begin{bmatrix} \frac{HL}{2cs^2EA} \\ \frac{-PL}{(1+2c^3)EA} \\ 0 \\ 0 \end{bmatrix}$$

Remark 1: It was named c = - sin (α) =-s and s= cos (α) =c.

$$\bar{u}_1 = T_1 u_1 = \begin{bmatrix} -\frac{HL}{2EAcs} - \frac{PLc}{EA(1+2c^3)} \\ \frac{PLs}{EA(1+2c^3)} - \frac{HL}{2EAs^2} \\ 0 \\ 0 \end{bmatrix}$$

Finally, the axial force will be:

$$F_1 = d_1 \frac{EA}{L} c = \frac{H}{2s} + \frac{Pc^2}{(1+2c^3)} \text{ (traction)}$$

Where

$$d_1 = 0 - \left(-\frac{HL}{2EAcs} - \frac{PLc}{EA(1+2c^3)}\right)$$

Axial Force in Bar 2

In this case the angle between bar 2 and the "x" axial is 90°. So, the matrix T_2 will be:

$$T_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad u_2 = \begin{bmatrix} \frac{HL}{2cs^2EA} \\ \frac{-PL}{(1+2c^3)EA} \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{u}_2 = T_2 u_2 = \begin{bmatrix} -\frac{PL}{EA(1+2c^3)} \\ -\frac{HL}{2EAcs^2} \\ 0 \\ 0 \end{bmatrix}$$

Finally, the axial force will be:

$$F_2 = d_2 \frac{EA}{L} c = \frac{Pc}{(1+2c^3)} \text{ (traction)}$$

Where

$$d_2 = 0 - \left(-\frac{PL}{EA(1+2c^3)}\right)$$

Axial Force in Bar 3

$$T_3 = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \quad u_3 = \begin{bmatrix} \dfrac{HL}{2cs^2EA} \\ \dfrac{-PL}{(1+2c^3)EA} \\ 0 \\ 0 \end{bmatrix}$$

Remark 1: It was named c = sin (α) = s and s= cos (α) =c.

$$\bar{u}_3 = T_3 u_3 = \begin{bmatrix} \dfrac{HL}{2EAcs} - \dfrac{PLc}{EA(1+2c^3)} \\ -\dfrac{HL}{2EAs^2} - \dfrac{PLs}{EA(1+2c^3)} \\ 0 \\ 0 \end{bmatrix}$$

Finally, the axial force will be:

$$F_3 = d_3 \frac{EA}{L} c = -\frac{H}{2s} + \frac{Pc^2}{(1+2c^3)}$$

Where

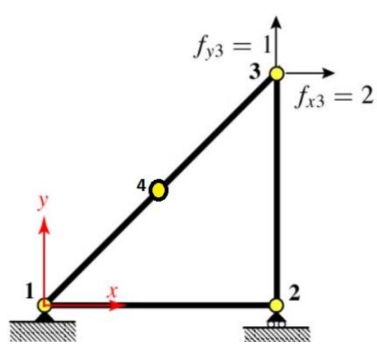
$$d_3 = 0 - \left(\frac{HL}{2EAcs} - \frac{PLc}{EA(1+2c^3)}\right)$$

In the limit case where α = 0 and $H \neq 0$, F_1 and F_3 tend to infinite due to the structure becomes in a mechanism where the nodes 2, 3 and 4 concur at the same node, in this case node 3. The existence of a horizontal force H produce an imbalance of moments ($\sum M \neq 0$) and a rotation around the node 3.

Assignment 2:

Dr. Who proposes "improving" the result for the example truss of the 1st lesson by putting one extra node, 4 at the midpoint of member (3) 1-3, so that it is subdivided in two different members: (3) 1-4 and (4) 3-4. His "reasoning" is that more is better. Try Dr. Who's suggestion by hand computations and verify that the solution "blows up" because the modified master stiffness is singular. Explain physically.

So, the scheme will be:



Taking into account all the consideration and remark that they had taken into account when the elemental stiffness matrices were computed, therefor it will proceed the compute the new elemental stiffness matrices.

Bar 1
The elemental stiffness matrix for bar 1:

$$K_1 = \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Bar 2
the elemental stiffness matrix for bar 2:

$$K_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Bar 3
The Bar 3 is a vertical bar, so, the elemental stiffness matrix is:

$$K_3 = \begin{bmatrix} 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -20 & -20 & 0 & 0 & 0 & 0 & 20 & 20 \\ -20 & -20 & 0 & 0 & 0 & 0 & 20 & 20 \end{bmatrix}$$

Bar 4
The Bar 4 is a vertical bar, so, the elemental stiffness matrix is:

$$K_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & 0 & -20 & -20 & 20 & 20 \\ 0 & 0 & 0 & 0 & -20 & -20 & 20 & 20 \end{bmatrix}$$

Assembly process

As the elemental stiffness matrices have been prepared for simplifying this process, then it is obtained the global stiffness matrix as $K_{glob} = K_1 + K_2 + K_3 + K_4$.

$$K_{glob} = \begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ -10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & -5 & 20 & 25 & -20 & -20 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \end{bmatrix}$$

After applying BC, the modified master stiffness will be:

$$\hat{K}_{glob} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 20 & 20 & -20 & -20 \\ 0 & 20 & 25 & -20 & -20 \\ 0 & -20 & -20 & 40 & 40 \\ 0 & -20 & -20 & 40 & 40 \end{bmatrix}$$

It can be seen that the matrix $\hat{K}_{glob}$ is singular (row 4 and 5 are linearly dependent or $\det(\hat{K}_{glob}) = 0$). So, the system of equation cannot be solved.

$$\begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 20 & 20 & -20 & -20 \\ 0 & 20 & 25 & -20 & -20 \\ 0 & -20 & -20 & 40 & 40 \\ 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The node added act as an articulation, so, the structure is behaved as mechanism.