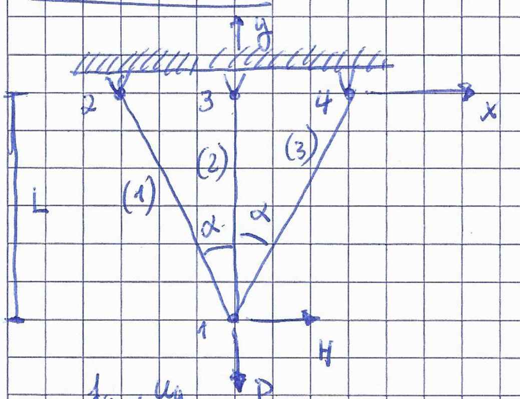
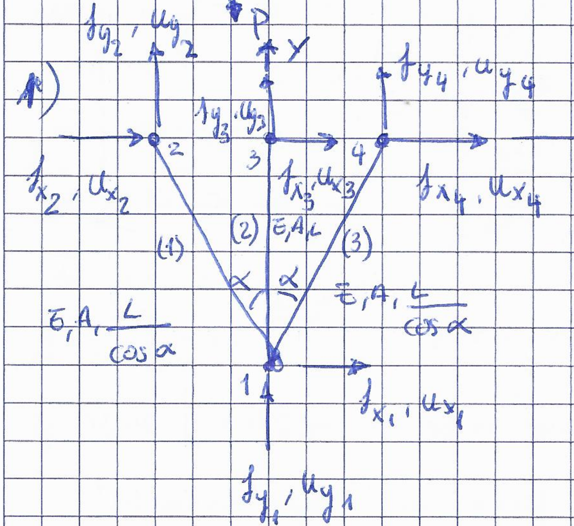
Assignment 1.1

17/02/2020

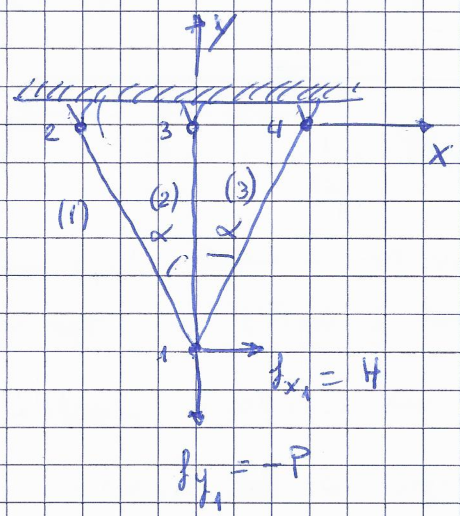


E, A same for all 3 bars.

1)



Applied loads



The structure has 4 nodes. Each one has 2 degrees of freedom, and so, the whole structure has 8 DOF's. The global stiffness matrix is 8 x 8. The global equation is: $f_{8\times1} = K_{8\times8} \times u_{8\times1}$, where:

$$f = [\, f_{x_1} \; f_{y_1} \; f_{x_2} \; f_{y_2} \; f_{x_3} \; f_{y_3} \; f_{x_4} \; f_{y_4} \,]^T$$

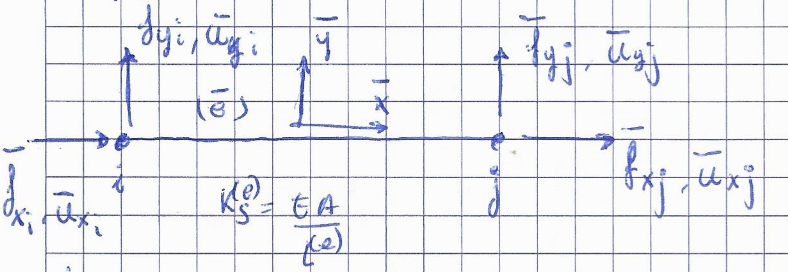
$$u = [\, u_{x_1} \; u_{y_1} \; u_{x_2} \; u_{y_2} \; u_{x_3} \; u_{y_3} \; u_{x_4} \; u_{y_4} \,]^T$$

$$K = \begin{bmatrix}
K_{x_1x_1} & K_{x_1y_1} & K_{x_1x_2} & K_{x_1y_2} & K_{x_1x_3} & K_{x_1y_3} & K_{x_1x_4} & K_{x_1y_4} \\
K_{y_1x_1} & K_{y_1y_1} & K_{y_1x_2} & K_{y_1y_2} & K_{y_1x_3} & K_{y_1y_3} & K_{y_1x_4} & K_{y_1y_4} \\
K_{x_2x_1} & K_{x_2y_1} & K_{x_2x_2} & K_{x_2y_2} & K_{x_2x_3} & K_{x_2y_3} & K_{x_2x_4} & K_{x_2y_4} \\
K_{y_2x_1} & K_{y_2y_1} & K_{y_2x_2} & K_{y_2y_2} & K_{y_2x_3} & K_{y_2y_3} & K_{y_2x_4} & K_{y_2y_4} \\
K_{x_3x_1} & K_{x_3y_1} & K_{x_3x_2} & K_{x_3y_2} & K_{x_3x_3} & K_{x_3y_3} & K_{x_3x_4} & K_{x_3y_4} \\
K_{y_3x_1} & K_{y_3y_1} & K_{y_3x_2} & K_{y_3y_2} & K_{y_3x_3} & K_{y_3y_3} & K_{y_3x_4} & K_{y_3y_4} \\
K_{x_4x_1} & K_{x_4y_1} & K_{x_4x_2} & K_{x_4y_2} & K_{x_4x_3} & K_{x_4y_3} & K_{x_4x_4} & K_{x_4y_4} \\
K_{y_4x_1} & K_{y_4y_1} & K_{y_4x_2} & K_{y_4y_2} & K_{y_4x_3} & K_{y_4y_3} & K_{y_4x_4} & K_{y_4y_4}
\end{bmatrix}$$

So as to build up the global equation, we need to obtain first the elemental stiffness matrices. The procedure is the following:

From theory*:

1) Consider a bar of area A, length L, and Young modulus E. For that bar, the elemental stiffness matrix is written according to the nodal forces and displacements. Localizing, we have: $\bar{f} = \bar{\bar{K}}\,\bar{u}$ $\Downarrow$



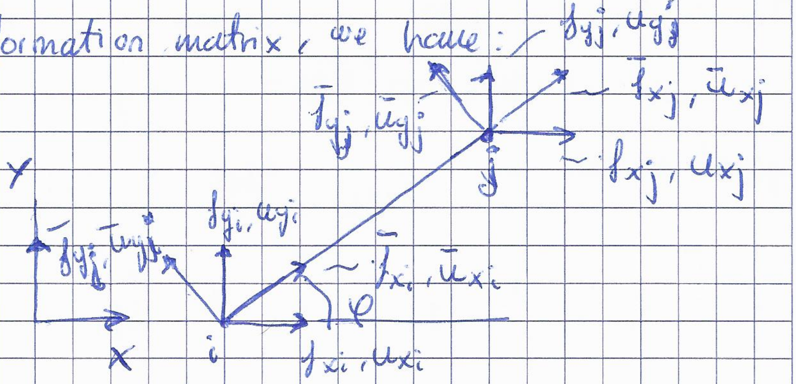
$$\begin{bmatrix} \bar{f}_{x_1} \\ \bar{f}_{y_1} \\ \bar{f}_{x_2} \\ \bar{f}_{y_2} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{x_1} \\ \bar{u}_{y_1} \\ \bar{u}_{x_2} \\ \bar{u}_{y_2} \end{bmatrix}$$

↳ the bar has 2 nodes (i,j) and each node has 2 DOF's. Thus, the stiffness matrix is 4x4.

$F = K_s^{e} d$ $\quad\Big|\quad d = \bar{u}_{xj} - \bar{u}_{xi}$, $\quad F = \bar{f}_{xj} = -\bar{f}_{xi}$

2) if we now consider a 2-noded element bar with an arbitrary angle φ with respect to the x-axis, the nodal forces and displacements will now need to get transformed to the standard / reference coordinate system. Using the proper transformation matrix, we have:

$$T^{(e)} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$



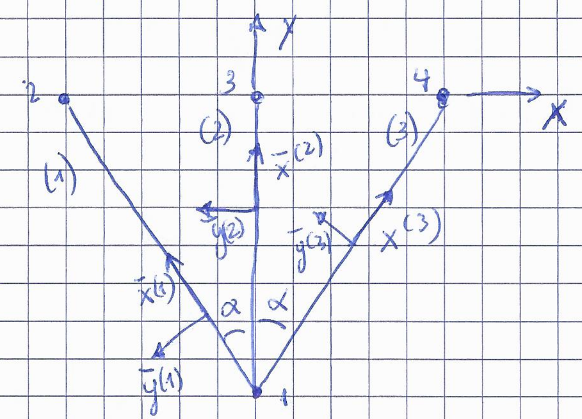
where $c = \cos\varphi$ and $\bar{u}^{(e)} = T^{(e)} u^{(e)}$

$s = \sin\varphi$ $\qquad f^{(e)} = (T^{(e)})^T \bar{f}^{(e)}$

And, globalizing for each element:

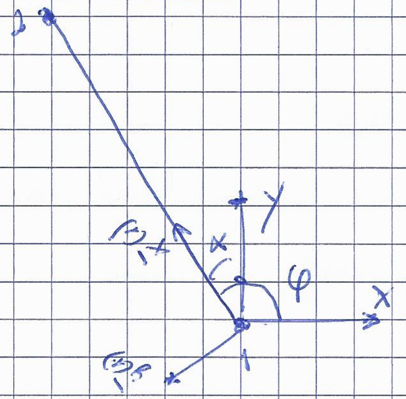
$$(T^{(e)})^T \bar{k}^{(e)} T^{(e)} = \frac{E^{(e)} A^{(e)}}{L^{(e)}} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

In our problem, we have:



For the first element

Trigonometric identities

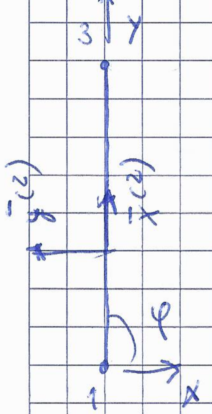


$$\varphi^{(1)} = \frac{\pi}{2} + \alpha \Rightarrow \begin{cases} \cos\varphi = \cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha \\ \sin\varphi = \sin\left(\frac{\pi}{2}+\alpha\right) = \cos\alpha \end{cases}$$

$$L^{(1)} = \frac{L}{\cos\alpha}$$

$$K^{(1)} = \frac{EA\cos\alpha}{L}\begin{bmatrix} s^2 & -sc & -s^2 & sc \\ -sc & c^2 & sc & -c^2 \\ -s^2 & sc & s^2 & -sc \\ sc & -c^2 & -sc & c^2 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} cs^2 & -sc^2 & -s^2c & sc^2 \\ -sc^2 & c^3 & sc^2 & -c^3 \\ -cs^2 & sc^2 & cs^2 & -sc^2 \\ sc^2 & -c^3 & -sc^2 & c^3 \end{bmatrix}$$

For the second element

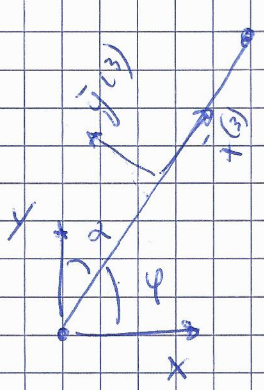


$$\varphi^{(2)} = \frac{\pi}{2} \Rightarrow \begin{cases} \cos\varphi = 0 \\ \sin\varphi = 1 \end{cases}$$

$$L^{(2)} = L$$

$$K^{(2)} = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

For the <u>third element</u>



$$\varphi^{(3)} = \frac{\pi}{2} - \alpha \Rightarrow \begin{cases} \cos\varphi = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha \\ \sin\varphi = \sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha \end{cases}$$

$$L^{(3)} = \frac{L}{\cos\alpha}$$

$$K^{(3)} = \frac{EA\cos\alpha}{L}\begin{bmatrix} s^2 & cs & -s^2 & -cs \\ cs & c^2 & -cs & -c^2 \\ -s^2 & -cs & s^2 & cs \\ -cs & -c^2 & cs & c^2 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} cs^2 & c^2s & -cs^2 & -c^2s \\ c^2s & c^3 & -c^2s & -c^3 \\ -cs^2 & -c^2s & cs^2 & c^2s \\ -c^2s & -c^3 & c^2s & c^3 \end{bmatrix}$$

if we write now the expanded element stiffness equations and then reconnect members by enforcing compatibility rule, we have:

$$\begin{bmatrix} f_{x_1}^{(1)} \\ f_{y_1}^{(1)} \\ f_{x_2}^{(1)} \\ f_{y_2}^{(1)} \\ f_{x_3}^{(1)} \\ f_{y_3}^{(1)} \\ f_{x_4}^{(1)} \\ f_{y_4}^{(1)} \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} cs^2 & -sc^2 & -s^2c & sc^2 & 0 & 0 & 0 & 0 \\ -sc^2 & c^3 & sc^2 & -c^3 & 0 & 0 & 0 & 0 \\ -cs^2 & sc^2 & cs^2 & -sc^2 & 0 & 0 & 0 & 0 \\ sc^2 & -c^3 & -sc^2 & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}$$

$$
\begin{bmatrix} f_{x_1}^{(2)} \\ f_{y_1}^{(2)} \\ f_{x_2}^{(2)} \\ f_{y_2}^{(2)} \\ f_{x_3}^{(2)} \\ f_{y_3}^{(2)} \\ f_{x_4}^{(2)} \\ f_{y_4}^{(2)} \end{bmatrix}
= \frac{EA}{L}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}
$$

$$
\begin{bmatrix} f_{x_1}^{(3)} \\ f_{y_1}^{(3)} \\ f_{x_2}^{(3)} \\ f_{y_2}^{(3)} \\ f_{x_3}^{(3)} \\ f_{y_3}^{(3)} \\ f_{x_4}^{(3)} \\ f_{y_4}^{(3)} \end{bmatrix}
= \frac{EA}{L}
\begin{bmatrix}
cs^2 & c^2s & 0 & 0 & 0 & 0 & -cs^2 & -c^2s \\
c^2s & c^3 & 0 & 0 & 0 & 0 & -c^2s & -c^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-cs^2 & -c^2s & 0 & 0 & 0 & 0 & cs^2 & c^2s \\
-c^2s & -c^3 & 0 & 0 & 0 & 0 & c^2s & c^3
\end{bmatrix}
\begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}
$$

if we now apply the equilibrium rule, we have:

$$
\underline{f} = \underline{f}^{(1)} + \underline{f}^{(2)} + \underline{f}^{(3)} = (\underline{\underline{k}}^{(1)} + \underline{\underline{k}}^{(2)} + \underline{\underline{k}}^{(3)})\underline{u} = \underline{\underline{K}}\,\underline{u}
$$

the global stiffness equation reads as:

$$
\begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_2} \\ f_{y_2} \\ f_{x_3} \\ f_{y_3} \\ f_{x_4} \\ f_{y_4} \end{bmatrix} = \frac{EA}{L}
\begin{bmatrix}
2cs^2 & 0 & -s^2c & sc^2 & 0 & 0 & -cs^2 & -c^2s \\
0 & 1+2c^3 & sc^2 & -c^3 & 0 & -1 & -c^2s & -c^3 \\
-cs^2 & sc^2 & cs^2 & -sc^2 & 0 & 0 & 0 & 0 \\
-sc^2 & -c^3 & -sc^2 & c^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
-cs^2 & -c^2s & 0 & 0 & 0 & 0 & cs^2 & c^2s \\
-c^2s & -c^3 & 0 & 0 & 0 & 0 & c^2s & c^3
\end{bmatrix}
\begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}
$$

the 5th row and column belong to the horizontal displacement of node 3. the horizontal forces are taken by the nodes 1 and 4. Meaning by this that node 3 ~~does~~ cannot hold any horizontal reaction and therefore it cannot move in horizontal direction.

2) As only forces are applied in node 1, the boundary conditions and applied loads yield as:

· Displacements BC $\longrightarrow$ $u_{x_2} = u_{y_2} = u_{x_3} = u_{y_3} = u_{x_4} = u_{y_4} = 0$

· Forces BC $\longrightarrow$ $f_{x_1} = H$

$f_{y_1} = -P$

if we apply these BC and remove the rows and columns belonging to known displacements, the reduced stiffness equation yields as:

$$
\begin{bmatrix} H \\ -P \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \end{bmatrix}
$$

3) $u_{x_1} = \dfrac{L}{EA} \dfrac{H}{2cs^2}$ $\qquad u_{y_1} = -\dfrac{L}{EA} \dfrac{P}{(1+2c^3)}$

3) Continue.

- when $\alpha \to 0 \Rightarrow c \to 1,\ s \to 0 \Rightarrow u_{x_1} \to \infty,\ u_{y_1} \to \frac{-P}{3}\frac{L}{EA}$

u_{x_1} "blows up" for $H \neq 0$ because the structure becomes a system of vertical bars in that same configuration (position). Since one of these two applied forces is horizontal and all the bars are ~~pin~~ ~~jointed~~ pinned, we are inducing a rotation in the system without causing or generating any bending moment. Therefore, the system can freely rotate for any $H \neq 0$.

- when $\alpha \to \frac{\pi}{2} \Rightarrow c \to 0,\ s \to 1 \Rightarrow u_{x_1} \to \infty,\ u_{y_1} \to -P\frac{L}{EA}$

Here, the bars 1 and 3 cannot compensate the moment generated by H. This is mainly due to tend the length of these two elements to infinity and thus, their stiffness is reduced to 0.

4) we can work the ~~eq~~ axial forces out from the equilibrium of forces at node 1 and the partial solution already given

$$\begin{cases} F_1 s - F_3 s = H \\ F_1 c + F_2 + F_3 c = P \\ F_3 = \dfrac{-H}{2s} + \dfrac{Pc^2}{1+2c^3} \end{cases}$$

with positive values for forces if the bar is submitted to tractions.

~~or~~ or just applying the following procedure for each element.

First element

1) $$\overline{u}^{(1)} = T^{(1)} u^{(1)} \Rightarrow \overline{u}^{(1)} = \begin{bmatrix} -s & c & 0 & 0 \\ -c & -s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{bmatrix} \begin{bmatrix} \frac{L}{EA}\frac{H}{2cs^2} \\ -\frac{L}{EA}\frac{P}{(1+2c^3)} \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{L}{EA}\begin{bmatrix} -\frac{H}{2cs} - \frac{Pc}{1+2c^3} \\ -\frac{H}{2cs} + \frac{Ps}{1+2c^3} \\ 0 \\ 0 \end{bmatrix}$$

2) $$d^{(1)} = \overline{u}_{x_2}^{(1)} - \overline{u}_{x_1}^{(1)} = 0 - \left(\frac{L}{EA}\left(-\frac{H}{2cs} - \frac{Pc}{1+2c^3}\right)\right) = \frac{L}{EA}\left(\frac{H}{2cs} + \frac{Pc}{1+2c^3}\right)$$

3) $$F^{(1)} = \frac{EA}{L/c} d^{(1)} = \frac{EA}{L/c}\left[\frac{L}{EA}\left(\frac{H}{2cs} + \frac{Pc}{1+2c^3}\right)\right] = \frac{H}{2s} + \frac{Pc^2}{1+2c^3}$$

Second element

1) $$\overline{u}^{(2)} = T^{(2)} u^{(2)} \Rightarrow \overline{u}^{(2)} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} \frac{L}{EA}\frac{H}{2cs^2} \\ -\frac{L}{EA}\frac{P}{(1+2c^3)} \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{L}{EA}\begin{bmatrix} -\frac{P}{1+2c^3} \\ -\frac{H}{2cs^2} \\ 0 \\ 0 \end{bmatrix}$$

2) $$d^{(2)} = \overline{u}_{x_2}^{(2)} - \overline{u}_{x_1}^{(2)} = 0 - \left(\frac{L}{EA}\left(-\frac{P}{1+2c^3}\right)\right) = \frac{L}{EA}\frac{P}{(1+2c^3)}$$

3) $$F^{(2)} = \frac{EA}{L} d^{(2)} = \frac{\cancel{EA}}{\cancel{L}}\frac{\cancel{L}}{\cancel{EA}}\frac{P}{(1+2c^3)} = \frac{P}{1+2c^3}$$

Third element

1) $\bar{u}^{(3)} = T^{(3)} u^{(3)} \Rightarrow \bar{u}^{(3)} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} \frac{L}{EA} \frac{H}{2cs^2} \\ -\frac{L}{EA} \frac{P}{(1+2c^3)} \\ 0 \\ 0 \end{bmatrix}$

$$= \frac{L}{EA} \begin{bmatrix} \frac{H}{2cs} - \frac{Pc}{1+2c^3} \\ -\frac{H}{2cs} - \frac{Ps}{1+2c^3} \\ 0 \\ 0 \end{bmatrix}$$

2) $d^{(3)} = \bar{u}_{x_2}^{(3)} - \bar{u}_{x_1}^{(3)} = 0 - \left(\frac{L}{EA} \left(\frac{H}{2cs} - \frac{Pc}{1+2c^3} \right) \right) = \frac{L}{EA} \left(\frac{Pc}{1+2c^3} - \frac{H}{2cs} \right)$

3) $F^{(3)} = \frac{EA}{L/c} d^{(3)} = \frac{\cancel{EA}}{\cancel{L}/c} \left(\frac{\cancel{L}}{\cancel{EA}} \left(\frac{Pc}{1+2c^3} - \frac{H}{2cs} \right) \right) = \frac{Pc^2}{1+2c^3} - \frac{H}{2s}$

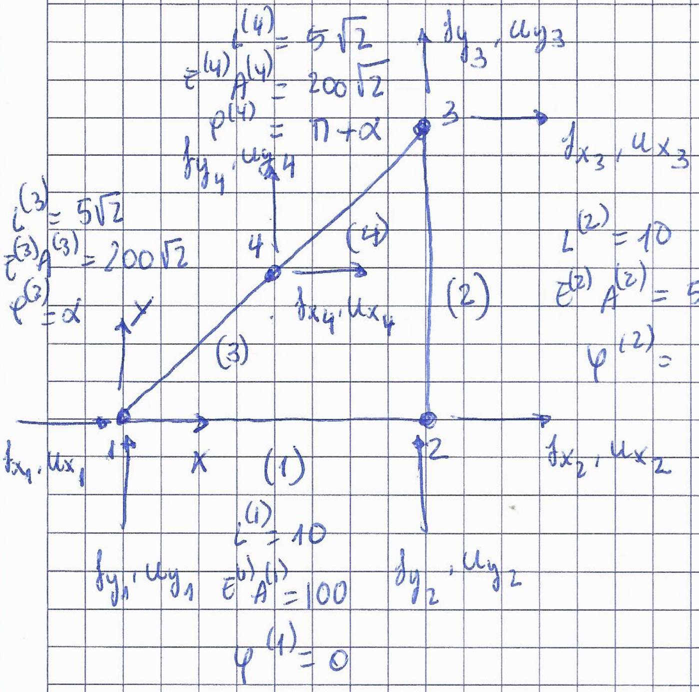
Rearranging the three axial forces:

$F^{(1)} = \frac{H}{2s} + \frac{Pc^2}{1+2c^3} \qquad F^{(2)} = \frac{P}{1+2c^3} \qquad F^{(3)} = -\frac{H}{2s} + \frac{Pc^2}{1+2c^3}$

If $H \neq 0$ and $\alpha \to 0 \Rightarrow s \to 0$ and since the term is in the denominator, as $\alpha \to 0$, $F^{(1)}$ and $F^{(3)} \longrightarrow \infty$.

The physical explanation resides in that ~~as the~~ the system in this case cannot hold equilibrium since no other axial force or restriction can compensate the bending moment ~~of a~~ generated by an applied force of $H \neq 0$.

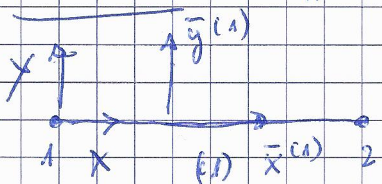
5) $\Rightarrow$ Assignment 1.2



The globalized element stiffness matrix is then:

$$K^{(1)} = \begin{bmatrix} 10 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad K^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 5 \end{bmatrix}$$

First element $\Uparrow$



$\varphi^{(1)} = 0$

$E^{(1)}A^{(1)} = 100$

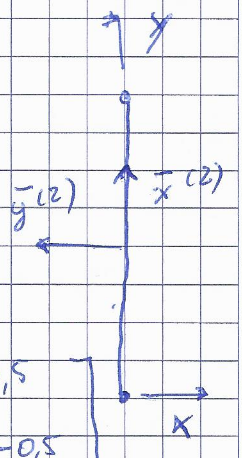
$L^{(1)} = 10$

Second element $\Uparrow$

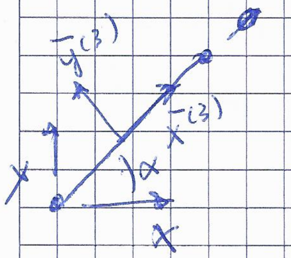
$\varphi^{(2)} = \frac{\pi}{2}$

$E^{(2)}A^{(2)} = 50$

$L^{(2)} = 10$



Third element



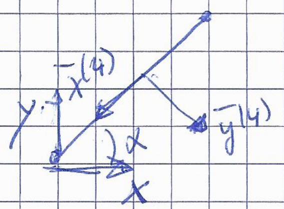
$\varphi^{(3)} = \alpha$

$E^{(3)}A^{(3)} = 200\sqrt{2}$

$L^{(3)} = 5\sqrt{2}$

$$K^{(3)} = 40\begin{bmatrix} 0,5 & 0,5 & -0,5 & -0,5 \\ 0,5 & 0,5 & -0,5 & -0,5 \\ -0,5 & -0,5 & 0,5 & 0,5 \\ -0,5 & -0,5 & 0,5 & 0,5 \end{bmatrix}$$

Fourth element



$$\varphi^{(4)} = \pi + \alpha \quad \begin{cases} s = \sin(\alpha + \pi) = -\sin\alpha \\ c = \cos(\alpha + \pi) = -\cos\alpha \end{cases}$$

$$E^{(4)} A^{(4)} = 200\sqrt{2}$$

$$l^{(4)} = 5\sqrt{2}$$

$$k^{(4)} = 40 \begin{bmatrix} 0{,}5 & 0{,}5 & -0{,}5 & -0{,}5 \\ 0{,}5 & 0{,}5 & -0{,}5 & -0{,}5 \\ -0{,}5 & -0{,}5 & 0{,}5 & 0{,}5 \\ -0{,}5 & -0{,}5 & 0{,}5 & 0{,}5 \end{bmatrix}$$

thus, from there, the expanded element stiffness equations leads to:

$$\begin{bmatrix} f_{x_1}^{(1)} \\ f_{y_1}^{(1)} \\ f_{x_2}^{(1)} \\ f_{y_2}^{(1)} \\ f_{x_3}^{(1)} \\ f_{y_3}^{(1)} \\ f_{x_4}^{(1)} \\ f_{y_4}^{(1)} \end{bmatrix} = \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}$$

$$\begin{bmatrix} f_{x_1}^{(2)} \\ f_{y_1}^{(2)} \\ f_{x_2}^{(2)} \\ f_{y_2}^{(2)} \\ f_{x_3}^{(2)} \\ f_{y_3}^{(2)} \\ f_{x_4}^{(2)} \\ f_{y_4}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}$$

$$\begin{bmatrix} f_{x_1}^{(3)} \\ f_{y_1}^{(3)} \\ f_{x_2}^{(3)} \\ f_{y_2}^{(3)} \\ f_{x_3}^{(3)} \\ f_{y_3}^{(3)} \\ f_{x_4}^{(3)} \\ f_{y_4}^{(3)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & 0 & -20 & -20 & 20 & 20 \\ 0 & 0 & 0 & 0 & -20 & -20 & 20 & 20 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}$$

$$\begin{bmatrix} f_{x_1}^{(4)} \\ f_{y_1}^{(4)} \\ f_{x_2}^{(4)} \\ f_{y_2}^{(4)} \\ f_{x_3}^{(4)} \\ f_{y_3}^{(4)} \\ f_{x_4}^{(4)} \\ f_{y_4}^{(4)} \end{bmatrix} = \begin{bmatrix} 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -20 & -20 & 0 & 0 & 0 & 0 & 20 & 20 \\ -20 & -20 & 0 & 0 & 0 & 0 & 20 & 20 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}$$

Now, after applying the equilibrium rule, the stiffness equation yields:

$$\begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_2} \\ f_{y_2} \\ f_{x_3} \\ f_{y_3} \\ f_{x_4} \\ f_{y_4} \end{bmatrix} = \begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ -10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & -5 & 20 & 25 & -20 & -20 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}$$

the boundary conditions are:

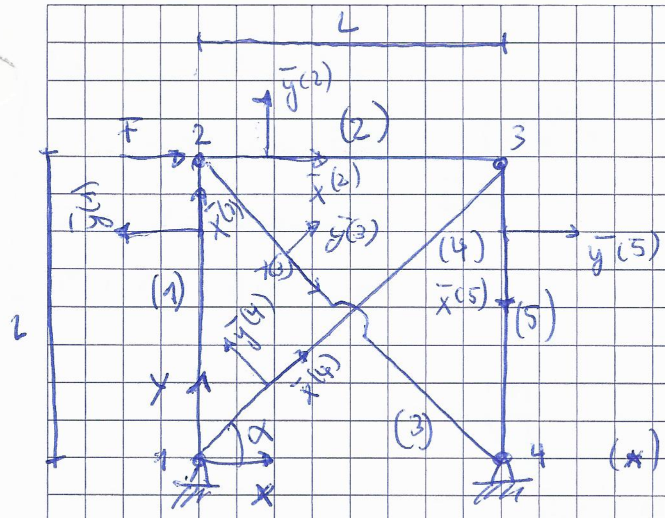
- Displacements BC $\rightarrow$ $u_{x_1} = u_{y_1} = u_{y_2} = 0$
- Forces BC $\rightarrow$ $\begin{cases} f_{x_2} = 0 \\ f_{x_3} = 2 \\ f_{y_3} = 1 \end{cases}$

the reduced stiffness equation, after applying the boundary conditions and deleting rows and columns belonging to known displacements yields as follows:

$$\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 20 & 20 & -20 & -20 \\ 0 & 20 & 25 & -20 & -20 \\ 0 & -20 & -20 & 40 & 40 \\ 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

the matrix is singular (the rank is lower than the matrix dimension since rows 4th and 5th are equal) and so, the problem is not well-posed. this is mainly because the system is under constrained and it leads to a mechanism of pinned bars. Therefore, the structure is unstable.

17/02/2020



$L = 6\ m$

$A = 6\ cm^2 \rightarrow 6 \cdot 10^{-4}\ m^2$

$E = 200\ GPa$

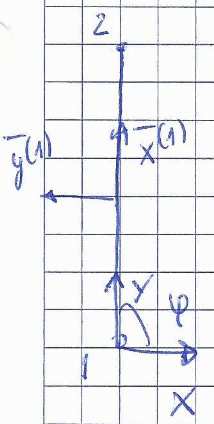
$\rightarrow EA = 120 \cdot 10^3\ kN$ $\downarrow$ Remains constant

$F = 80\ kN$

(*) we know that in an 2-noded bar with an arbitrary φ angle:

$$K^e = \frac{E^e A^e}{L^e}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

For the first element



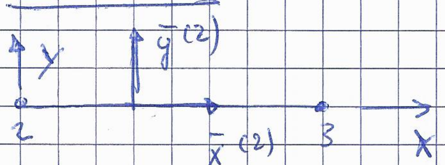
$$\varphi^{(1)} = \frac{\pi}{2} \Rightarrow \begin{cases} \cos\frac{\pi}{2} = 0 \\ \sin\frac{\pi}{2} = 1 \end{cases}$$

$E^{(1)}A^{(1)} = 120 \cdot 10^3\ kN$

$L^{(1)} = 6\ m$

$$K^{(1)} = \frac{120 \cdot 10^3}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Second element



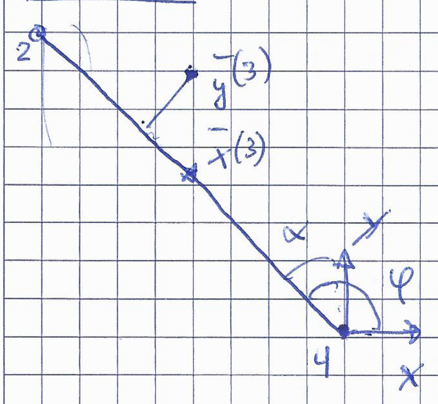
$$\varphi^{(2)} = 0 \begin{cases} \cos 0 = 1 \\ \sin 0 = 0 \end{cases}$$

$E^{(2)}A^{(2)} = 120 \cdot 10^3\ kN$

$L^{(2)} = 6\ m$

$$K^{(2)} = \frac{120 \cdot 10^3}{6}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Third element



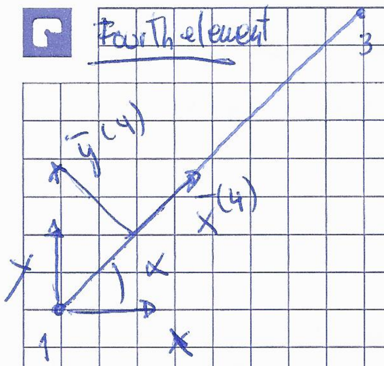
$$\varphi^{(3)} = \alpha - \frac{\pi}{2} \Rightarrow \begin{cases} s = \sin\left(\alpha - \frac{\pi}{2}\right) = -\cos\alpha \\ c = \cos\left(\alpha - \frac{\pi}{2}\right) = \sin\alpha \end{cases}$$

$L^{(3)} = 6\sqrt{2}\ m$

$E^{(3)}A^{(3)} = 120 \cdot 10^3\ kN$

$$K^{(3)} = \frac{120 \cdot 10^3}{6\sqrt{2}}\begin{bmatrix} 0,5 & -0,5 & -0,5 & 0,5 \\ -0,5 & 0,5 & 0,5 & -0,5 \\ -0,5 & 0,5 & 0,5 & -0,5 \\ 0,5 & -0,5 & -0,5 & 0,5 \end{bmatrix}$$

Fourth element



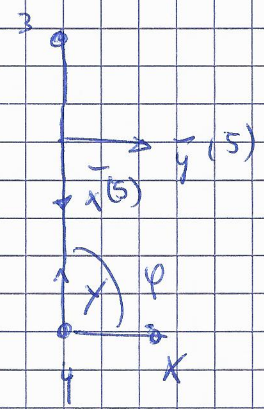
$$\varphi^{(4)} = \alpha \quad \begin{cases} s = \sin\alpha \\ c = \cos\alpha \end{cases}$$

$$E^{(4)} A^{(4)} = 120 \cdot 10^3 \ kN$$

$$l^{(4)} = 6\sqrt{2} \ m$$

$$k^{(4)} = \frac{120 \cdot 10^3}{6\sqrt{2}} \begin{bmatrix} 0,5 & 0,5 & -0,5 & -0,5 \\ 0,5 & 0,5 & -0,5 & -0,5 \\ -0,5 & -0,5 & 0,5 & 0,5 \\ -0,5 & -0,5 & 0,5 & 0,5 \end{bmatrix}$$

Fifth element



$$\varphi^{(5)} = -\frac{\pi}{2} \Rightarrow \begin{cases} s = \sin\left(-\frac{\pi}{2}\right) = -1 \\ c = \cos\left(-\frac{\pi}{2}\right) = 0 \end{cases}$$

$$E^{(5)} A^{(5)} = 120 \cdot 10^3 \ kN$$

$$l^{(5)} = 6 m$$

$$k^{(5)} = \frac{120 \cdot 10^3}{6} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

the expanded element stiffness equations read after reconnecting members by enforcing the compatibility rule as:

$$\begin{bmatrix} f_{x_1}^{(1)} \\ f_{y_1}^{(1)} \\ f_{x_2}^{(1)} \\ f_{y_2}^{(1)} \\ f_{x_3}^{(1)} \\ f_{y_3}^{(1)} \\ f_{x_4}^{(1)} \\ f_{y_4}^{(1)} \end{bmatrix} = \frac{120 \cdot 10^3}{6} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}$$

$$
\begin{bmatrix} f_{x_1}^{(2)} \\ f_{y_1}^{(2)} \\ f_{x_2}^{(2)} \\ f_{y_2}^{(2)} \\ f_{x_3}^{(2)} \\ f_{y_3}^{(2)} \\ f_{x_4}^{(2)} \\ f_{y_4}^{(2)} \end{bmatrix}
= \frac{120 \cdot 10^3}{6}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}
$$

$$
\begin{bmatrix} f_{x_1}^{(3)} \\ f_{y_1}^{(3)} \\ f_{x_2}^{(3)} \\ f_{y_2}^{(3)} \\ f_{x_3}^{(3)} \\ f_{y_3}^{(3)} \\ f_{x_4}^{(3)} \\ f_{y_4}^{(3)} \end{bmatrix}
= \frac{120 \cdot 10^3}{6\sqrt{2}}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0{,}5 & -0{,}5 & 0 & 0 & -0{,}5 & 0{,}5 \\
0 & 0 & -0{,}5 & 0{,}5 & 0 & 0 & 0{,}5 & -0{,}5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0{,}5 & 0{,}5 & 0 & 0 & 0{,}5 & -0{,}5 \\
0 & 0 & 0{,}5 & -0{,}5 & 0 & 0 & -0{,}5 & 0{,}5
\end{bmatrix}
\begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}
$$

$$
\begin{bmatrix} f_{x_1}^{(4)} \\ f_{y_1}^{(4)} \\ f_{x_2}^{(4)} \\ f_{y_2}^{(4)} \\ f_{x_3}^{(4)} \\ f_{y_3}^{(4)} \\ f_{x_4}^{(4)} \\ f_{y_4}^{(4)} \end{bmatrix}
= \frac{120 \cdot 10^3}{6\sqrt{2}}
\begin{bmatrix}
0{,}5 & 0{,}5 & 0 & 0 & -0{,}5 & -0{,}5 & 0 & 0 \\
0{,}5 & 0{,}5 & 0 & 0 & -0{,}5 & -0{,}5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0{,}5 & -0{,}5 & 0 & 0 & 0{,}5 & 0{,}5 & 0 & 0 \\
-0{,}5 & -0{,}5 & 0 & 0 & 0{,}5 & 0{,}5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}
$$

$$\begin{bmatrix} f_{x_1}^{(5)} \\ f_{y_1}^{(5)} \\ f_{x_2}^{(5)} \\ f_{y_2}^{(5)} \\ f_{x_3}^{(5)} \\ f_{y_3}^{(5)} \\ f_{x_4}^{(5)} \\ f_{y_4}^{(5)} \end{bmatrix} = \frac{120 \cdot 10^{3}}{6} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix}$$

And so, after applying the equilibrium rule, the stiffness equation yields,

$$\begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_2} \\ f_{y_2} \\ f_{x_3} \\ f_{y_3} \\ f_{x_4} \\ f_{y_4} \end{bmatrix} = \begin{bmatrix} 0{,}7071 & 0{,}7071 & 0 & 0 & -0{,}7071 & -0{,}7071 & 0 & 0 \\ 0{,}7071 & 2{,}7071 & 0 & -2 & -0{,}7071 & -0{,}7071 & 0 & 0 \\ 0 & 0 & 2{,}7071 & -0{,}7071 & -2 & 0 & -0{,}7071 & 0{,}7071 \\ 0 & -2 & -0{,}7071 & 2{,}7071 & 0 & 0 & 0{,}7071 & -0{,}7071 \\ -0{,}7071 & -0{,}7071 & -2 & 0 & 2{,}7071 & 0{,}7071 & 0 & 0 \\ -0{,}7071 & -0{,}7071 & 0 & 0 & 0{,}7071 & 2{,}7071 & 0 & -2 \\ 0 & 0 & -0{,}7071 & 0{,}7071 & 0 & 0 & 0{,}7071 & -0{,}7071 \\ 0 & 0 & 0{,}7071 & -0{,}7071 & 0 & -2 & -0{,}7071 & 2{,}7071 \end{bmatrix} \times 1e{+}04$$

$$\cdot \begin{bmatrix} u_{x_1} & u_{y_1} & u_{x_2} & u_{y_2} & u_{x_3} & u_{y_3} & u_{x_4} & u_{y_4} \end{bmatrix}^T$$

If we now apply the boundary conditions such that:

Forces BC's $\rightarrow \begin{cases} f_{x_2} = 80 \text{ KN} \\ f_{y_2} = f_{x_3} = f_{y_3} = 0 \end{cases}$

Displacements BC's $\rightarrow u_{x_1} = u_{y_1} = u_{x_4} = u_{y_4} = 0$

The reduced stiffness equation, deleting the rows and columns belonging to known displacements yields as follows:

$$\begin{bmatrix} 80 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2{,}7071 & -0{,}7071 & -2 & 0 \\ -0{,}7071 & 2{,}7071 & 0 & 0 \\ -2 & 0 & 2{,}7071 & 0{,}7071 \\ 0 & 0 & 0{,}7071 & 2{,}7071 \end{bmatrix} \times 1 \cdot 10^4 \times \begin{bmatrix} u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \end{bmatrix}$$

Solving for the unknowns:

$$\begin{bmatrix} u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \end{bmatrix} = \begin{bmatrix} 0{,}00854139 \\ 0{,}002231029 \\ 0{,}00677242 \\ -0{,}00176897 \end{bmatrix} [m]$$