Computational Structural Mechanics and Dynamics

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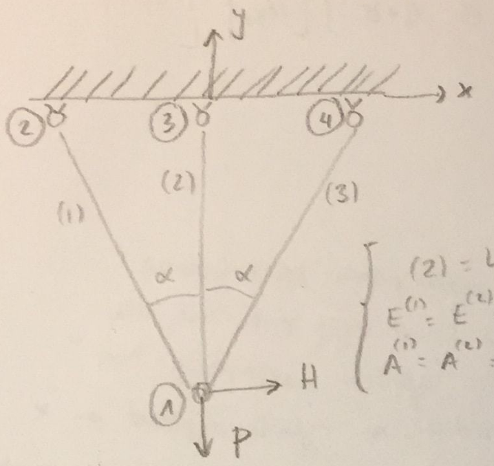
Master in Num. Methods
2018

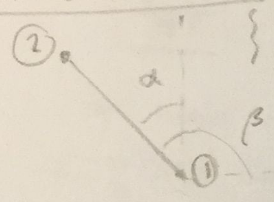
Assignment 1 :



$$\begin{cases} (2) = L \\ E^{(1)} = E^{(2)} = E^{(3)} \\ A^{(1)} = A^{(2)} = A^{(3)} \end{cases}$$

$$k^{(e)} = \frac{E^e A^e}{L^e}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

For element 1 :

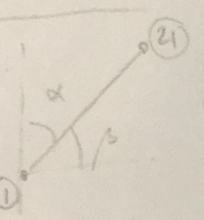


$$\begin{cases} \sin\beta = \cos\alpha \\ \cos\beta = -\sin\alpha \end{cases}$$

$$\begin{array}{c} \quad u_{x1} \quad u_{y2} \quad\quad u_{x2} \quad u_{y2} \end{array}$$

$$k^{(1)} = \frac{EA}{L}\begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -sc & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix}$$

For element 3 :

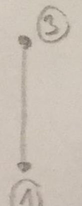


$$\begin{cases} \cos\beta = \sin\alpha \\ \sin\beta = \cos\alpha \end{cases}$$

$$\begin{array}{c} \quad u_{x1} \quad u_{y1} \quad u_{x4} \quad u_{y4} \end{array}$$

$$k^{(3)} = \frac{EA}{L}\begin{bmatrix} s^2 & sc & -s^2 & -sc \\ sc & c^2 & -sc & -c^2 \\ -s^2 & -cs & s^2 & cs \\ -cs & -c^2 & cs & c^2 \end{bmatrix}$$

For element 2 :



$c = 0$

$$\begin{array}{c} \quad u_{x1} \quad u_{y1} \quad u_{x3} \quad u_{y3} \end{array}$$

$$k^{(2)} = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

If we construct the actual matrix of stiffness we get:

$$[K] = \frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -s^2c & c^2s & 0 & 0 & -cs^2 & -sc^2 \\ 0 & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -sc^2 & -c^3 \\ -cs^2 & c^2s & cs^2 & -sc^2 & 0 & 0 & 0 & 0 \\ c^2s & -c^3 & -sc^2 & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -cs^2 & -c^2s & 0 & 0 & 0 & 0 & cs^2 & sc^2 \\ -c^2s & -c^3 & 0 & 0 & 0 & 0 & sc^2 & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} H \\ -P \\ R_{x2} \\ R_{y2} \\ R_{x3} \\ R_{y3} \\ R_{x4} \\ R_{y4} \end{bmatrix}$$

Now we have a 2×2 system of equations that after solving it we get: (Independent variables)

$$\begin{cases} 2cs^2 \cdot u_{x_1} = H \\ 2c^3 H \cdot u_{y_1} = -P \end{cases}$$

$$\boxed{u_{x_1} = \frac{H}{2cs^2}\,\frac{L}{EA}}$$

$$\boxed{u_{y_1} = \frac{-P}{1+2c^3}\,\frac{L}{EA}}$$

The system is solved by:
$$\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^2 \end{bmatrix}\begin{bmatrix} u_{x_1} \\ u_{y_1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

We test for $\boxed{\alpha \to 0}$

$$\underset{\alpha\to 0}{u_{x_1}} = \frac{H}{0}\,\frac{L}{EA} \quad \text{and} \quad \underset{\alpha\to 0}{u_{y_1}} = \frac{-P}{3}\,\frac{L}{EA}$$

, this solution makes sense because there's no angle and thus, the force applied horizontally "H" produces an infinite displacement on "x". The vertical displacement u_{y_1} is constant: $\boxed{\frac{-P}{3}\frac{L}{EA}}$

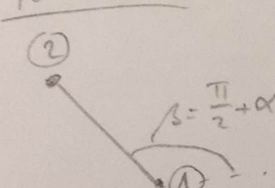
We test for $\boxed{\alpha \to \frac{\pi}{2}}$

$$\underset{\alpha\to \pi/2}{u_{x_1}} = \frac{H}{0}\,\frac{L}{EA}, \quad \underset{\alpha\to \frac{\pi}{2}}{u_{y_1}} = \frac{-PL}{0EA}$$

The horizontal displacement u_{x_1} blows up because there's no elements supporting the horizontal load H. It works fine for P.

We recover the axial forces in the 3 members:

For element 1:



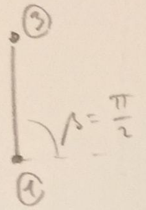
$$u^{(1)} = \begin{bmatrix} u_{x_1} & u_{y_1} & u_{x_2} & u_{y_2} \end{bmatrix}^T = \begin{bmatrix} \frac{H}{2cs^2} & \frac{-P}{1+2c^3} & 0 & 0 \end{bmatrix}^T \frac{L}{AE}$$

$$\bar{u}^{(1)} = T^{(1)} u^{(1)} = \begin{bmatrix} -s & c & 0 & 0 \\ -c & -s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{bmatrix}\begin{bmatrix} \frac{H}{2cs^2} \\ -\frac{P}{1+2c^3} \\ 0 \\ 0 \end{bmatrix}\frac{L}{AE} = \frac{L}{AE}\begin{bmatrix} \frac{-H}{2cs} - \frac{Pc}{1+2c^3} \\ \frac{-H}{2s^2} + \frac{Ps}{1+2c^3} \\ 0 \\ 0 \end{bmatrix}$$

$$d^{(1)} = \bar{u}_{x_2}^{(1)} - \bar{u}_{x_1}^{(1)} = 0 + \left(\frac{H}{2cs} + \frac{Pc}{1+2c^3}\right)\frac{L}{AE}$$

$$\boxed{F_1 = \frac{AE}{L}\left(\frac{H}{2s} + \frac{Pc^2}{1+2c^3}\right)}$$

Axial forces for element 2:

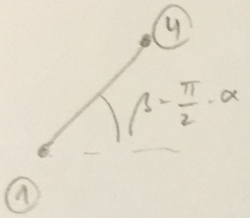


$$u_2 = \left[u_{x_1} \; u_{y_1} \; u_{x_3} \; u_{y_3}\right]^T = \left[\frac{H}{2cs^2} \quad \frac{-P}{1+2c^3} \quad 0 \quad 0\right]^T \frac{L}{AE}$$

$$\bar{u}^{(2)} = T^{(2)} \cdot u^{(2)} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} H/2cs^2 \\ -P/1+2c^3 \\ 0 \\ 0 \end{bmatrix} \frac{L}{AE} = \begin{bmatrix} \frac{H}{2s^2} - \frac{Ps}{1+2c^3} \\ \frac{-H}{2cs} + \frac{Pc}{1+2c^3} \\ 0 \\ 0 \end{bmatrix} \frac{L}{AE}$$

$$d^{(2)} = \bar{u}^{(2)}_{x_3} - \bar{u}^{(2)}_{x_1} = \frac{L}{AE}\left(\frac{-H}{2s^2} + \frac{Ps}{1+2c^3}\right) \qquad \boxed{f^{(2)} = \frac{AE}{L}\left(\frac{H}{2s^2} + \frac{Ps}{1+2c^3}\right)}$$

Axial forces for element 3:



$$u^{(3)} = \left[u_{x_1} \; u_{y_1} \; u_{x_4} \; u_{y_4}\right]^T = \left[\frac{H}{2cs^2} \quad \frac{\cdot P}{1+2c^3} \quad 0 \quad 0\right]^T \frac{L}{AE}$$

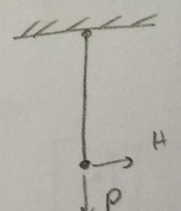
$$\bar{u}^{(3)} = T^{(3)} \cdot u^{(3)} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} H/2cs^2 \\ -P/1+2c^3 \\ 0 \\ 0 \end{bmatrix} \frac{L}{AE} = \begin{bmatrix} \frac{H}{2cs} - \frac{Pc}{1+2c^3} \\ \frac{-H}{2c^2} - \frac{Ps}{1+2c^3} \\ 0 \\ 0 \end{bmatrix} \frac{L}{AE}$$

$$d^{(3)} = \bar{u}^{(3)}_{x_4} - u^{(3)}_{x_1} = \frac{L}{AE}\left(\frac{-H}{2cs} + \frac{Pc}{1+2c^3}\right) \qquad \boxed{F^{(3)} = \frac{AE}{L}\left(\frac{-H}{2s} + \frac{Pc^2}{1+2c^3}\right)}$$

If we look at the axial force when $H \neq 0$ and $\alpha \to 0$

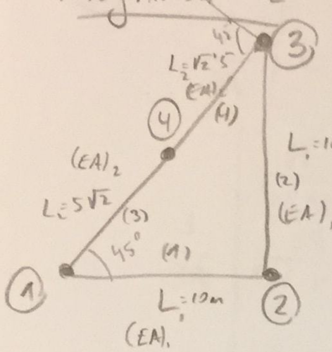
we have that:

$$\left.\begin{aligned} F^{(1)} &= \frac{H}{0} + \frac{P}{3} \\ F^{(3)} &= \frac{H}{0} + \frac{P}{3} \end{aligned}\right\}$$



The truss is not fixed, so the axial forces will "blow up" due to the lack of boundary. (Infinite Rotation)

Assignment 2.



$L_1 = 10\,m$

$(EA)_1 = 50$

$(EA)_2 = 200\sqrt{2}$

For Element 1

$$\frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \end{bmatrix} = \begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_2} \\ f_{y_2} \end{bmatrix}$$

Applying BC:

$$10\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_{x_1}^{(1)} \\ u_{y_1}^{(1)} \\ u_{x_2}^{(1)} \\ u_{y_2}^{(1)} \end{bmatrix} = \begin{bmatrix} f_{x_1}^{(1)} \\ f_{y_1}^{(1)} \\ f_{x_2}^{(1)} \\ f_{y_2}^{(1)} \end{bmatrix}$$

For Element 2:

$$\frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \end{bmatrix} = \begin{bmatrix} f_{x_2} \\ f_{y_2} \\ f_{x_3} \\ f_{y_3} \end{bmatrix} \xrightarrow{\text{we get}} 5\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \end{bmatrix} = \begin{bmatrix} f_{x_2} \\ f_{y_2} \\ f_{x_3} \\ f_{y_3} \end{bmatrix}$$

For element 3:

$$\frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_4} \\ u_{y_4} \end{bmatrix} = \begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_4} \\ f_{y_4} \end{bmatrix} \longrightarrow \frac{200\sqrt{2}}{5\sqrt{2}}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1/2 & 1/2 \\ -\frac{1}{2} & -\frac{1}{2} & 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_4} \\ u_{y_4} \end{bmatrix} = \begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_4} \\ f_{y_4} \end{bmatrix}$$

For element 4:

$$\frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix} = \begin{bmatrix} f_{x_3} \\ f_{y_3} \\ f_{x_4} \\ f_{y_4} \end{bmatrix} \longrightarrow \frac{200\sqrt{2}}{5\sqrt{2}}\begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/\sqrt{2} \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix} = \begin{bmatrix} f_{x_3} \\ f_{y_3} \\ f_{x_4} \\ f_{y_4} \end{bmatrix}$$

Knowing that $\boxed{f^{(e)} = K^{(e)} \cdot u^{(e)}}$ we can assemble the General/global stiffness matrix of the truss such as:

$$f = f^{(1)} + f^{(2)} + f^{(3)} + f^{(4)} = \left(K^{(1)} + K^{(2)} + K^{(3)} + K^{(4)}\right) u$$

$$\begin{array}{cccccccc} u_{x_1} & u_{y_1} & u_{x_2} & u_{y_2} & u_{x_3} & u_{y_3} & u_{x_4} & u_{y_4} \end{array}$$
$$\begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ -10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & 0 & 20 & 25 & -20 & -20 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \end{bmatrix}\begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix} = \begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_2} \\ f_{y_2} \\ f_{x_3} \\ f_{y_3} \\ f_{x_4} \\ f_{y_4} \end{bmatrix}$$

The Boundary conditions on the problem state that

$$\boxed{u_{x_1} = u_{y_1} = u_{y_2} = 0}$$

The known forces by the problem are $\begin{cases} f_{x_3} = 2 \\ f_{y_3} = 1 \end{cases}$

So now we can construct the new system matrix:

(5)

$$\begin{array}{c} \\ \\ \\ R_4 \rightarrow \\ R_5 \rightarrow \end{array}\begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 20 & 20 & -20 & -20 \\ 0 & 20 & 25 & -20 & -20 \\ 0 & -20 & -20 & 40 & 40 \\ 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

We can now observe that Row 4 and Row 5 are the same, this means the determinant of the matrix will be 0. $\det|K| = 0 \rightarrow$ Singular. When we splited the element in 2, we added a unnecessary instability to the system. Resulting in a confusing final result.