Master's Degree Numerical Methods in Engineering

Universitat Politècnica de Catalunya
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Computational Structural Mechanics and Dynamics

Homework 1: Direct Stiffness Method

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Contents

Abstract

In this homework two structures were analysed using the Direct Stiffness Method. The structures were a porch and a truss. For the porch structure, the displacement in the unrestrained nodes were calculated. For the truss structure, a generic solution was implemented leaving all the results in terms of cross section, Young's Modulus, length and angle of the bars. A general analysis of the behaviour of the structure was performed for different angles. Finally, a truss structure was modified by adding a node in the middle of an existing bar, and it resulted in a non-determined problem as the displacement has to be a linear combination of the existing node.

1 Introduction

The direct stiffness method is composed by two blocks, first the '*Breakdown*' composed of disconnection of the elements, localization of axis, numbering of nodes and members and finally, members formations in which the elements are classified according to their geometric characteristics. The second block of actions are called '*Assembly and Solution*' in which the system is globalized (the elements are drafted again in their original positions and the equivalent geometric entities are establish), afterwards the members are merged again, the boundary conditions applied and the problem is solved. Post-processing actions are undertake afterwards.

For solving problems of two-noded bars in 2D movements (x and y directions), firstly the element stiffness matrix are calculated and then the global problem can be solved after the assembly. The problem is solved using the *displacement-based* scheme for each node of the system. The local problem is shown in equation (1).

$$\overline{f}^{(e)} = \overline{K}^{(e)} \cdot \overline{u}^{(e)} \tag{1}$$

$$\begin{bmatrix} \overline{f}_{xi} \\ \overline{f}_{yi} \\ \overline{f}_{xj} \\ \overline{f}_{yj} \end{bmatrix} = \begin{bmatrix} \overline{K}_{xixi} & \overline{K}_{xiyi} & \overline{K}_{xixj} & \overline{K}_{xiyj} \\ \overline{K}_{yixi} & \overline{K}_{yiyi} & \overline{K}_{yixj} & \overline{K}_{yiyj} \\ \overline{K}_{xjxi} & \overline{K}_{xjyi} & \overline{K}_{xjxj} & \overline{K}_{xjyj} \\ \overline{K}_{yjxi} & \overline{K}_{yjyi} & \overline{K}_{yjxj} & \overline{K}_{yjyj} \end{bmatrix} \cdot \begin{bmatrix} \overline{u}_{xi} \\ \overline{u}_{yi} \\ \overline{u}_{xj} \\ \overline{u}_{yj} \end{bmatrix}$$

The local stiffness matrix is calculated considering the axial stiffness of the bar as a spring, where the force is proportional to the axial deformation $(u_{xj} - u_{xi})$:

$$\begin{bmatrix} \overline{K}_{xixi} & \overline{K}_{xiyi} & \overline{K}_{xixj} & \overline{K}_{xiyj} \\ \overline{K}_{yixi} & \overline{K}_{yiyi} & \overline{K}_{yixj} & \overline{K}_{yiyj} \\ \overline{K}_{xjxi} & \overline{K}_{xjyi} & \overline{K}_{xjxj} & \overline{K}_{xjyj} \\ \overline{K}_{yjxi} & \overline{K}_{yjyi} & \overline{K}_{yjxj} & \overline{K}_{yjyj} \end{bmatrix} = \frac{EA}{L} \cdot \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is important to notice that as the strains are calculated using the displacement of the nodes, a formulation considering the orientation of the bars is of keen importance to find the solution of the problem. The general formulation is presented in Figure 1.

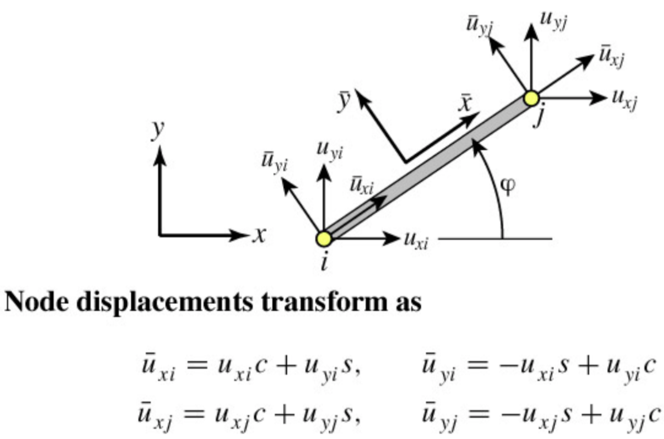

Figure 1: General formulation of displacement based strains.

where c and s are $cos(\phi)$ and $sin(\phi)$ respectively.

For transforming a '*global*' u_{xi} displacement to a '*local*' $\overline{u}_{xi}$ the expression above can be written in matrix notation as:

$$\begin{bmatrix} \overline{u}_{xi} \\ \overline{u}_{yi} \\ \overline{u}_{xj} \\ \overline{u}_{yj} \end{bmatrix} = \begin{bmatrix} cos(\phi) & sin(\phi) & 0 & 0 \\ -sin(\phi) & cos(\phi) & 0 & 0 \\ 0 & 0 & cos(\phi) & sin(\phi) \\ 0 & 0 & -sin(\phi) & cos(\phi) \end{bmatrix} \cdot \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix}$$

Then, the expression can be written as:

$$\overline{u}^{(e)} = T^{(e)} \cdot u^{(e)} \tag{2}$$

An equivalent transformation can be written for the applied forces, from global to local coordinates, leading to the same expression written above in the form

$$\overline{f}^{(e)} = T^{(e)} \cdot f^{(e)} \tag{3}$$

Finally, the global stiffness matrix takes the form:

$$K^{(e)} = (T^{(e)})^T \cdot \overline{K}^{(e)} \cdot T^{(e)} \tag{4}$$

$$K^{(e)} = \frac{E^{(e)} A^{(e)}}{L^{(e)}} \cdot \begin{bmatrix} cos^2(\phi) & sin(\phi)\ cos(\phi) & -cos^2(\phi) & -sin(\phi)\ cos(\phi) \\ & sin^2(\phi) & -sin(\phi)\ cos(\phi) & -sin^2(\phi) \\ & & cos^2(\phi) & sin(\phi)\ cos(\phi) \\ & sym. & & sin^2(\phi) \end{bmatrix}$$

2 Problem resolution

2.1 Problem 1 - Classwork

The first structure is a plane porch subjected to a lateral load in the top as shown in Figure 2. Both beams and columns from the structure have a length of $L = 6.0\ m$, cross section of $A = 6.0\ cm^2$, Young's modulus of $E = 200\ GPa$ and the applied force is $F = 80.0\ kN$.

The structure is given already as an idealized FEM structure, with its nodes and bars numbered and local axis fixed. Therefore, the structure is ready to be calculated. The local stiffness matrices are calculated for each element using equation (4).

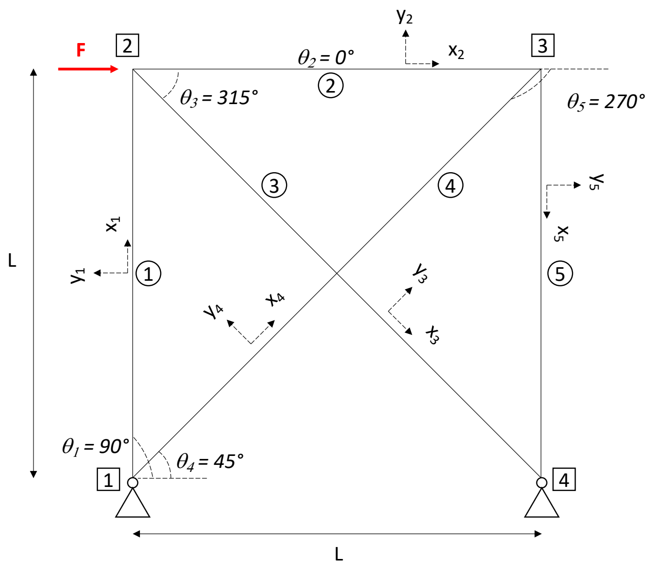

Figure 2: Porch structure.

<u>Element 1 ($\theta = 90°$)</u>

$$K^{(1)} = \frac{EA}{L^{(1)}} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

<u>Element 2 ($\theta = 0°$)</u>

$$K^{(2)} = \frac{EA}{L^{(2)}} \cdot \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Element 3 ($\theta = 315°$)</u>

$$K^{(3)} = \frac{EA}{L^{(3)}} \cdot \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

<u>Element 4 ($\theta = 45°$)</u>

$$K^{(4)} = \frac{EA}{L^{(4)}} \cdot \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}$$

Element 5 ($\theta = 270°$)

$$K^{(5)} = \frac{EA}{L^{(5)}} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Finally, assembling the system we get to:

$$K_G\ [N/m] = \begin{bmatrix} 7071068 & 7071068 & 0 & 0 & -7071068 & -7071068 & 0 & 0 \\ 7071068 & 27071068 & 0 & -20000000 & -7071068 & -7071068 & 0 & 0 \\ 0 & 0 & 27071068 & -7071068 & -20000000 & 0 & -7071068 & 7071068 \\ 0 & -20000000 & -7071068 & 27071068 & 0 & 0 & 7071068 & -7071068 \\ -7071068 & -7071068 & -20000000 & 0 & 27071068 & 7071068 & 0 & 0 \\ -7071068 & -7071068 & 0 & 0 & 7071068 & 27071068 & 0 & -20000000 \\ 0 & 0 & -7071068 & 7071068 & 0 & 0 & -10000000 & -10000000 \\ 0 & 0 & 7071068 & -7071068 & 0 & -20000000 & -10000000 & 10000000 \end{bmatrix}$$

Afterwards, the force vector and boundary conditions are applied as follows:

$$f_G\ [N] = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 80000\ N \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$u_G\ [m] = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ 0 \\ 0 \end{bmatrix}$$

And the system is then reduced to:

$$K_G^* \cdot u_G^* = f_G^*$$

$$\begin{bmatrix} 27071068 & -7071068 & -20000000 & 0 \\ -7071068 & 27071068 & 0 & 0 \\ -20000000 & 0 & 27071068 & 7071068 \\ 0 & 0 & 7071068 & 27071068 \\ -7071068 & 7071068 & 0 & 0 \\ 7071068 & -7071068 & 0 & -20000000 \end{bmatrix} N/m \cdot \begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 80000N \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We find the following solution for the system:

$$\begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 8.54E-03 \\ 2.23E-03 \\ 6.77E-03 \\ -1.77E-03 \end{bmatrix} m$$

2.2 Problem 2 - Homework - Assingnment 1

Consider the truss problem defined in the Figure 3. All geometric and material properties: L, α, E and A, as well as the applied forces P and H are to be kept as variables. This truss has 8 degrees of freedom, with six of them removable by the fixed-displacement conditions at nodes 2, 3 and 4. This structure is statically indeterminate as long as $\alpha \neq 0$.

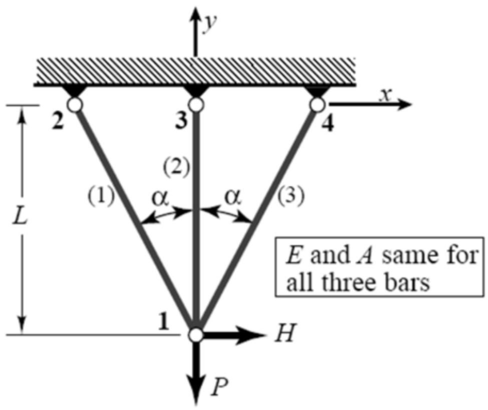

Figure 3: Truss structure.

1. Show that the master stiffness equations are:

$$\frac{EA}{L}\cdot\begin{bmatrix} 2c(\alpha)s^2(\alpha) & 0 & -c(\alpha)s^2(\alpha) & -c^2(\alpha)s(\alpha) & 0 & 0 & -c(\alpha)s^2(\alpha) & -c^2(\alpha)s(\alpha) \\ & 1+2c^3(\alpha) & c^2(\alpha)s(\alpha) & -c^3(\alpha) & 0 & -1 & -c^2(\alpha)s(\alpha) & -c^3(\alpha) \\ & & c(\alpha)s^2(\alpha) & -c^2(\alpha)s(\alpha) & 0 & 0 & 0 & 0 \\ & & & c^3(\alpha) & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & c(\alpha)s^2(\alpha) & c^2(\alpha)s(\alpha) \\ symm & & & & & & & c^3(\alpha) \end{bmatrix}\cdot\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}=\begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $c(\alpha)$ is $cos(\alpha)$ and $s((\alpha)$ is $sin((\alpha)$. Explain from physics why the fifth row and column contain only zeros.

2. Apply the BCs and show the 2-equation modified stiffness system.

3. Solve for the displacements u_{x1} and u_{y1}. Check that the solution makes physical sense for the limit cases $\alpha \to 0$ and $\alpha \to \pi/2$.

4. Recover the axial forces in the three elements. Partial answer: $F^{(3)} = -H/(2s(\alpha)) + Pc^2(\alpha)/(1 + 2c^3(\alpha))$. Why do $F^{(1)}$ and $F^{(3)}$ "blow up" of $H \neq 0$ and $\alpha \to 0$?

2.2.1 Master stiffness equations

From Figure 3 the angles for each of the bars are shown in Table 1.

Element	Angle
(1)	90°+α
(2)	90°
(3)	90°-α

Table 1: Bars orientation.

Using Equation (4) again for the orientations shown previously and remembering the following trigonometric properties:

$$sin(90^\circ - \alpha) = cos(\alpha)$$

$$cos(90° - \alpha) = sin(\alpha)$$
$$sin(90° + \alpha) = cos(\alpha)$$
$$sin(90° - \alpha) = -sin(\alpha)$$

The length of the members (1) and (3) depend on the angle α therefore it has to be treated as an unknown, considering the following equation:

$$L_{(1)} = L_{(3)} = \frac{L}{cos(\alpha)}$$

Element 1 ($\theta = 90° + \alpha$)

$$K^{(1)} \cdot \overline{u}^{(1)} = \frac{EA}{L} \cdot \begin{bmatrix} c(\alpha)s^2(\alpha) & -c^2(\alpha)s(\alpha) & -c(\alpha)s^2(\alpha) & c^2(\alpha)s(\alpha) \\ & c^3(\alpha) & c^2(\alpha)s(\alpha) & -c^3(\alpha) \\ & & c(\alpha)s^2(\alpha) & -c^2(\alpha)s(\alpha) \\ & & & c^3(\alpha) \end{bmatrix} \cdot \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix}$$

Element 2 ($\theta = 90°$)

$$K^{(2)} \cdot \overline{u}^{(2)} = \frac{EA}{L} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{x1}^{(2)} \\ u_{y1}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{bmatrix}$$

Element 3 ($\theta = 90° - \alpha$)

$$K^{(3)} \cdot \overline{u}^{(3)} = \frac{EA}{L} \cdot \begin{bmatrix} c(\alpha)s^2(\alpha) & c^2(\alpha)s(\alpha) & -c(\alpha)s^2(\alpha) & -c^2(\alpha)s(\alpha) \\ & c^3(\alpha) & -c^2(\alpha)s(\alpha) & -c^3(\alpha) \\ & & c(\alpha)s^2(\alpha) & c^2(\alpha)s(\alpha) \\ & & & c^3(\alpha) \end{bmatrix} \cdot \begin{bmatrix} u_{x1}^{(3)} \\ u_{y1}^{(3)} \\ u_{x4}^{(3)} \\ u_{y4}^{(3)} \end{bmatrix}$$

With these expressions for each of the matrices, assembling the global system we get to the matrix given in the exercise. The reason why the $fifth$ row and column of the matrix is zero, is that taking into consideration that this row and column are the horizontal displacement of the node 3:

- if the node would register any movement it would mean that there is an horizontal force applied in it;
- if there is an horizontal force in this node, the force is applied externally (which is not the case) or the bar would have to support a bending moment, and therefore the structure is not a truss any more;
- finally, even if the structure was not fixed in the upper nodes, and a truss bar was join these nodes, the orthogonallity between the bars, would have given an unload vertical bar.

2.2.2 Apply BC

The displacements in nodes 2, 3 and 4 are known are equal to zero.

$$\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then the system of equations is reduced to:

$$\frac{EA}{L} \cdot \begin{bmatrix} 2c(\alpha)s^2(\alpha) & 0 \\ 0 & 1 + 2c^3(\alpha) \end{bmatrix} \cdot \begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

2.2.3 Solve for the displacements u_{x1} and u_{y1}

The displacements are obtained using the previous results as:

$$u_{x1} = \frac{H \cdot L}{EA \cdot 2cos(\alpha)sin^2(\alpha)}$$

$$u_{y1} = \frac{-P \cdot L}{EA \cdot (1 + 2 \cdot cos^3(\alpha))}$$

Check that the solution makes physical sense for the limit cases $\alpha \to 0$ and $\alpha \to \pi/2$

When $\alpha \to 0$ the system is collapsed into a single vertical bar, where, as truss structure do not have bending stiffness, the system would have an infinite displacement in the horizontal direction. In the vertical direction, three bars would give the system its stiffness.

$$u_{x1} = \frac{H \cdot L}{EA \cdot (2 \cdot (\to 1) \cdot (\to 0))} \to \infty$$

$$u_{y1} = \frac{-P \cdot L}{EA \cdot (1 + 2 \cdot (\to 1))} \to \frac{-P \cdot L}{3 \cdot EA}$$

When $\alpha \to \pi/2$ the system looks like an inverse 'T' and the vertical bar in the middle is the only one that can take the vertical force in node 1. Then the horizontal bars will have a length related with $L/cos(\alpha)$ and as $cos(\alpha) \to 0$ the horizontal length will tend to infinite, and therefore the axial stiffness will tend to zero $EA/(L/cos(\alpha)) \to 0$. The vertical displacement depends only on the stiffness of the vertical bar as expected.

$$u_{x1} = \frac{H \cdot L}{EA \cdot (2 \cdot (\to 0) \cdot (\to 1))} \to \infty$$

$$u_{y1} = \frac{-P \cdot L}{EA \cdot (1 + 2 \cdot (\to 0))} \to \frac{-P \cdot L}{EA}$$

Recover the axial forces in the three elements. Why do $\mathbf{F}^{(1)}$ and $\mathbf{F}^{(3)}$ "blow up" of $H \neq 0$ and $\alpha \to 0$

To find the axial forces in the elements, the stiffness matrix will be used replacing the displacements calculated previously and the forces on the fixities will be used to find the axial force.

<u>Element 1 ($\theta = 90° + \alpha$)</u>

$$\frac{EA}{L} \cdot \begin{bmatrix} c(\alpha)s^2(\alpha) & -c^2(\alpha)s(\alpha) & -c(\alpha)s^2(\alpha) & c^2(\alpha)s(\alpha) \\ & c^3(\alpha) & c^2(\alpha)s(\alpha) & -c^3(\alpha) \\ & & c(\alpha)s^2(\alpha) & -c^2(\alpha)s(\alpha) \\ & & & c^3(\alpha) \end{bmatrix} \cdot \begin{bmatrix} \frac{H \cdot L}{EA \cdot 2c(\alpha)s^2(\alpha)} \\ \frac{-P \cdot L}{EA \cdot (1+2 \cdot c^3(\alpha))} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix}$$

Using the equations on the third and fourth row:

$$f_{x2}^{(1)} = -\frac{H}{2} - \frac{P \cdot cos^2(\alpha)\ sin(\alpha)}{1 + 2\ cos^3(\alpha)}$$

$$f_{y2}^{(1)} = \frac{H \cdot cos(\alpha)}{2 \cdot sin(\alpha)} - \frac{P \cdot cos^3(\alpha)}{1 + 2\ cos^3(\alpha)}$$

Then the axial force in the element 1 is:

$$f_{ax}^{(1)} = \frac{f_{y1}^{(1)}}{cos(\alpha)} = \frac{H}{2sin(\alpha)} - \frac{P \cdot cos^2(\alpha)}{1 + 2\ cos^3(\alpha)}$$

Element 2 ($\theta = 90°$)

$$\frac{EA}{L} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{H \cdot L}{EA \cdot 2c(\alpha)s^2(\alpha)} \\ \frac{-P \cdot L}{EA \cdot (1+2 \cdot c^3(\alpha))} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_{x1}^{(2)} \\ f_{y1}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix}$$

As stated in the first task of the problem in this case the axial force will depend only on the vertical displacement:

$$f_{ax}^{(2)} = \frac{P}{1 + 2 \cdot cos^3(\alpha)}$$

Element 3 ($\theta = 90° - \alpha$)

$$\frac{EA}{L} \cdot \begin{bmatrix} c(\alpha)s^2(\alpha) & c^2(\alpha)s(\alpha) & -c(\alpha)s^2(\alpha) & -c^2(\alpha)s(\alpha) \\ & c^3(\alpha) & -c^2(\alpha)s(\alpha) & -c^3(\alpha) \\ & & c(\alpha)s^2(\alpha) & c^2(\alpha)s(\alpha) \\ & & & c^3(\alpha) \end{bmatrix} \cdot \begin{bmatrix} \frac{H \cdot L}{EA \cdot 2c(\alpha)s^2(\alpha)} \\ \frac{-P \cdot L}{EA \cdot (1+2 \cdot c^3(\alpha))} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x4}^{(3)} \\ f_{y4}^{(3)} \end{bmatrix}$$

Using the equations on the third and fourth row:

$$f_{x4}^{(3)} = -\frac{H}{2} + \frac{P \cdot cos^2(\alpha)\ sin(\alpha)}{1 + 2\ cos^3(\alpha)}$$

$$f_{y4}^{(3)} = -\frac{H \cdot cos(\alpha)}{2 \cdot sin(\alpha)} + \frac{P \cdot cos^3(\alpha)}{1 + 2\ cos^3(\alpha)}$$

Then the axial force in the element 3 is:

$$f_{ax}^{(3)} = \frac{f_{y1}^{(3)}}{cos(\alpha)} = -\frac{H}{2 \cdot sin(\alpha)} + \frac{P \cdot cos^2(\alpha)}{1 + 2\ cos^3(\alpha)}$$

As stated before, when $\alpha \to 0$ the bending stiffness of the bars cannot stand the horizontal force and this is seen in both first terms of axial forces of element (1) and (3), the division $H/(2 \cdot sin(\alpha) \to \infty$.

2.3 Problem 2 - Assignment 2

Dr. Who proposes "improving" the result for the example truss of the 1st lesson by putting one extra node, 4 at the midpoint of member (3) 1-3, so that it is subdivided in two different members: (3) 1-4 and (4) 3-4. His "reasoning" is that more is better. Try Dr. Who's suggestion by hand computations and verify that the solution "blows up" because the modified master stiffness is singular. Explain physically.

The exercise to solve is the one shown in the Figure 4, but adding a node in the middle between nodes 1 and 3. The element between nodes 1 and 2 is the element (1), between nodes 2 and 3 is the element (2) and between nodes 1 and the one to add, we call it 4, is the element (3) and the element between nodes 4 and 3 will be the element (4).

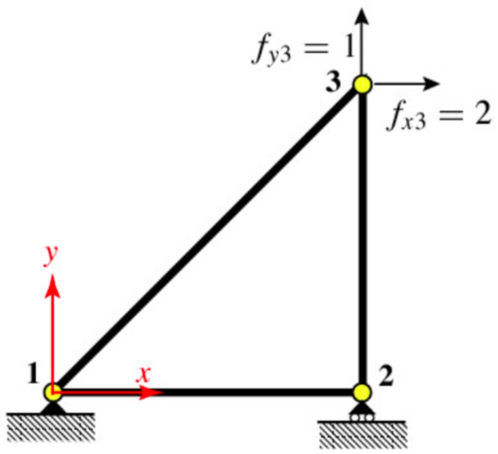

Figure 4: Structure to analize.

From the class, the stiffness matrix of elements (1) and (2) remain unchanged, and for the elements (3) and (4), only a change of length is needed. The matrices are:

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix} = 10 \cdot \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix}$$

$$\begin{bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} = 5 \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{bmatrix}$$

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x4}^{(3)} \\ f_{y4}^{(3)} \end{bmatrix} = 40 \cdot \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} u_{x1}^{(3)} \\ u_{y1}^{(3)} \\ u_{x4}^{(3)} \\ u_{y4}^{(3)} \end{bmatrix}$$

$$\begin{bmatrix} f_{x4}^{(4)} \\ f_{y4}^{(4)} \\ f_{x3}^{(4)} \\ f_{y3}^{(4)} \end{bmatrix} = 40 \cdot \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} u_{x4}^{(3)} \\ u_{y4}^{(3)} \\ u_{x3}^{(3)} \\ u_{y3}^{(3)} \end{bmatrix}$$

After assembling the global stiffness matrix:

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{bmatrix} = \begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ & & 10 & 0 & 0 & 0 & 0 & 0 \\ & & & 5 & 0 & -5 & 0 & 0 \\ & & & & 40 & 40 & -20 & -20 \\ & & & & & 40 & -20 & -20 \\ & & & & & & 40 & 40 \\ & & & & & & & 40 \end{bmatrix} \cdot \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

After applying the boundary conditions:

$$\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ & 40 & 40 & -20 & -20 \\ & & 40 & -20 & -20 \\ & & & 40 & 40 \\ & & & & 40 \end{bmatrix} \cdot \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

The solution of the problem is not unique as the matrix has determinant equal zero. Column 2 is equal to column 3 and column 4 is equal to column 5. This means that the node added in the structure cannot take any displacement value, but needs to be a linear combination of the displacement registered in node 3. Also means that there are infinite possible results that guarantee the equilibrium of the system.
The reason for the not unique displacement field is because the structure is a truss type, then the displacement in the loaded node depends on the loads and the stiffness of the bars reaching to that node. As there is no load applied to the node, nor any other change in the material or stiffness or geometry, requiring the addition of a node, the node do not provide any further information. The displacement of any point of the bar is a linear combination of the movement of the loaded node and the other edge of the bar (in this case the fixed node 1).