Computational Structural Mechanics and Dynamics

Assignment 1

On "The Direct Stiffness Method":

Consider the truss problem defined in the Figure. All geometric and material properties: L, α, E and A, as well as the applied forces P and H, are to be kept as variables. This truss has 8 degrees of freedom, with six of them removable by the fixed-displacement conditions at nodes 2, 3 and 4. This structure is statically indeterminate as long as $\alpha \neq 0$.

(a) Show that the master stiffness equations are

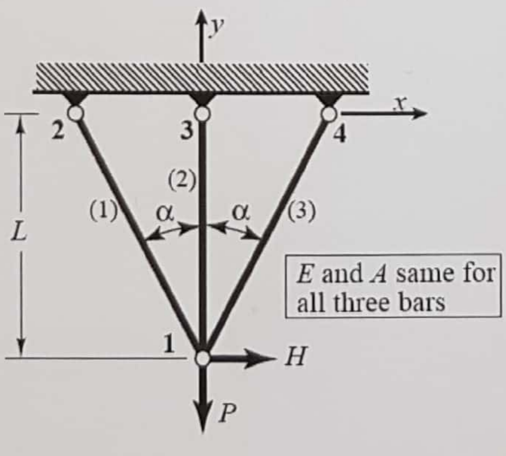
$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ \text{symm} & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

in which $c = \cos\alpha$ and $s = \sin\alpha$. Explain from physics why the 5th row and column contain only zeros.

(b) Apply the BCs and show the 2-equation modified stiffness system.

(c) Solve for the displacements u_{x1} and u_{y1}. Check that the solution makes physical sense for the limit cases $\alpha \to 0$ and $\alpha \to \pi/2$. Why does u_{x1} "blow up" if $H \neq 0$ and $\alpha \to 0$?

(d) Recover the axial forces in the three members. Partial answer: $F^{(3)} = -H/(2s) + Pc^2/(1+2c^3)$. Why do $F^{(1)}$ and $F^{(3)}$ "blow up" if $H \neq 0$ and $\alpha \to 0$?



Assignment 2

Dr. Who proposes "improving" the result for the example truss of the 1st lesson by putting one extra node, 4 at the midpoint of member (3) 1-3, so that it is subdivided in two different members: (3) 1-4 and (4) 3-4. His "reasoning" is that more is better. Try Dr. Who's suggestion by hand computations and verify that the solution "blows up" because the modified master stiffness is singular. Explain physically.

Date of Assignment: 10/02/2020
Date of Submission: 17/02/2020

The assignment must be submitted as a pdf file named **As1-Surname.pdf** to the CIMNE virtual center.

Computational Structural Mechanics & Dynamics.

As 1— Direct Stiffness Method

Kiran Kolhe

Prog. in Computational Mechanics

Assignment 1 -

a) Element ① → angle is $\frac{\pi}{2} + \alpha$

Element ② → angle is $\frac{\pi}{2}$

Element ③ → angle is $\frac{\pi}{2} - \alpha$

$$K^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

$$\sin\left(\frac{\pi}{2}+\alpha\right) = \sin\frac{\pi}{2}\cdot\cos\alpha + \cos\alpha\cdot\sin\frac{\pi}{2} = \cos\alpha$$

$$\sin\left(\frac{\pi}{2}-\alpha\right) = \sin\frac{\pi}{2}\cdot\cos\alpha - \cos\frac{\pi}{2}\cdot\sin\alpha = \cos\alpha$$

$$\cos\left(\frac{\pi}{2}+\alpha\right) = \cos\frac{\pi}{2}\cdot\cos\alpha - \sin\frac{\pi}{2}\cdot\sin\alpha = -\sin\alpha$$

$$\cos\left(\frac{\pi}{2}-\alpha\right) = \cos\frac{\pi}{2}\cdot\cos\alpha + \sin\frac{\pi}{2}\cdot\sin\alpha = \sin\alpha$$

* E, A for all the bars is same.

element ① -

$$\cos\alpha = \frac{L}{L^{(1)}} \rightarrow \therefore L^{(1)} = \frac{L}{c}. \quad (c = \cos\alpha).$$

$$K^{(1)} = \frac{EAc}{L} \begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix} \qquad f^{(1)} = \begin{bmatrix} f_{x_1}^{(1)} \\ f_{y_1}^{(1)} \\ f_{x_2}^{(1)} \\ f_{y_2}^{(1)} \end{bmatrix}$$

element ② -

$$L^{(2)} = L$$

~~cos~~

$$K^{(2)} = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad f^{(2)} = \begin{bmatrix} f_{x_1}^{(2)} \\ f_{y_2}^{(2)} \\ f_{x_3}^{(2)} \\ f_{y_3}^{(2)} \end{bmatrix}$$

element ③ -

$$\cos\alpha = \frac{L}{L^{(3)}} \rightarrow \therefore L^{(3)} = \frac{L}{c}$$

$$K^{(3)} = \frac{EAc}{L} \begin{bmatrix} s^2 & cs & -s^2 & -cs \\ cs & c^2 & -cs & -c^2 \\ -s^2 & -cs & s^2 & cs \\ -cs & -c^2 & cs & c^2 \end{bmatrix} \qquad f^{(3)} = \begin{bmatrix} f_{x_1}^{(3)} \\ f_{y_1}^{(3)} \\ f_{x_4}^{(3)} \\ f_{y_4}^{(3)} \end{bmatrix}$$

Now, the total dof of the system are 8. Hence, augmenting the elemental level matrices, the equ's become -

$$K^{(e)} u = f^{(e)}$$

$$K^{(1)} \cdot u = f^{(1)}$$

$$\frac{EA}{L}\begin{bmatrix} cs^2 & -c^2s & -cs^2 & c^2s & 0 & 0 & 0 & 0 \\ -c^2s & c^3 & c^2s & -c^3 & 0 & 0 & 0 & 0 \\ -cs^2 & c^2s & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ c^2s & -c^3 & -c^2s & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix} = \begin{bmatrix} f_{x_1}^{(1)} \\ f_{y_1}^{(1)} \\ f_{x_2}^{(1)} \\ f_{y_2}^{(1)} \\ f_{x_3}^{(1)} \\ f_{y_3}^{(1)} \\ f_{x_4}^{(1)} \\ f_{y_4}^{(1)} \end{bmatrix}$$

element ①

$$\frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix} = \begin{bmatrix} f_{x_1}^{(2)} \\ f_{y_1}^{(2)} \\ f_{x_2}^{(2)} \\ f_{y_2}^{(2)} \\ f_{x_3}^{(2)} \\ f_{y_3}^{(2)} \\ f_{x_4}^{(2)} \\ f_{y_4}^{(2)} \end{bmatrix}$$

element ②

$$\frac{EA}{L}\begin{bmatrix} cs^2 & c^2s & 0 & 0 & 0 & 0 & -cs^2 & -c^2s \\ c^2s & c^3 & 0 & 0 & 0 & 0 & -c^2s & -c^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -cs^2 & -c^2s & 0 & 0 & 0 & 0 & cs^2 & c^2s \\ -c^2s & -c^3 & 0 & 0 & 0 & 0 & c^2s & c^3 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix} = \begin{bmatrix} f_{x_1}^{(3)} \\ f_{y_1}^{(3)} \\ f_{x_2}^{(3)} \\ f_{y_2}^{(3)} \\ f_{x_3}^{(3)} \\ f_{y_3}^{(3)} \\ f_{x_4}^{(3)} \\ f_{y_4}^{(3)} \end{bmatrix}$$

element ③

Now,

$(K_1 + K_2 + K_3)\, u = (f_1 + f_2 + f_3)$ since, nodes 2, 3, 4 are fixed, nodal forces at these nodes are zero.

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ 0 & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ -cs^2 & c^2s & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ c^2s & -c^3 & -c^2s & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -cs^2 & -c^2s & 0 & 0 & 0 & 0 & cs^2 & c^2s \\ -c^2s & -c^3 & 0 & 0 & 0 & 0 & c^2s & c^3 \end{bmatrix}\begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This is the master stiffness equ. of the problem.

We can observe that the 5^{th} row & column of the master stiffness matrix is zero. It corresponds to the horizontal disp. of element ② at node 3. Physically, from the figure we see that element ② is aligned along vertical axis & is fixed at node 3 by pin joint. As, we know that, a truss member can be loaded only axially, & here, the member is vertical hence it can be loaded in Y dir. only. So, the horizontal or x-axis disp. for this element ② are zero. ~~A truss undergoes defo~~ Hence, the 5^{th} row & column are zero.

b) Applying BCs →

$$u_{x_2} = u_{y_2} = u_{x_3} = u_{y_3} = u_{x_4} = u_{y_4} = 0.$$

∴ Reduced sys. of equ. is -

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} u_{x_1} \\ u_{y_1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

c)

$$\therefore \quad 2cs^2\, u_{x_1} = \frac{HL}{EA} \rightarrow \therefore \boxed{u_{x_1} = \frac{HL}{2EA\,cs^2}}$$

$$(1+2c^3)\, u_{y_1} = \frac{-PL}{EA} \rightarrow \therefore \boxed{u_{y_1} = \frac{-PL}{EA(1+2c^3)}}$$

For $\alpha \to 0$, $u_{x_1} = \frac{HL}{0}$ $\therefore u_{x_1} \to \infty$.

& $u_{y_1} = \frac{-PL}{3EA}$.

- The sol. makes physical sense because as $\alpha = 0$, all the the 3 elements are in same direction & that can be considered as one bar only, with 3 times stiffness of each element.

i.e in u_{y_1} we can see, $u_{y_1} = \frac{-P}{3\left(\frac{EA}{L}\right)}$ → 3 times stiffness of 1 element $\left(\frac{EA}{L}\right)$.

- if $H \neq 0$ when $\alpha \to 0$, the sol. blows up because $u_{x_1} \to \infty$. The sol. remains undefined at u_{x_1} which probably means that it could not withstand any ~~forces or~~ moments at node 1.

For $\alpha \to \frac{\pi}{2}$,

$u_{x_1} = \frac{HL}{0}$ $\therefore u_{x_1} \to \infty$

& $u_{y_1} = -\frac{PL}{EA}$.

when α is $\frac{\pi}{2}$, elements ① & ③ become horizontal & ∥ to x-axis. And also, we know length of these 2 elements is given by $L^{(1)} = \frac{L}{c}$ & $L^{(3)} = \frac{L}{c}$, so when $\alpha \to \frac{\pi}{2}$, then $c \to 0$ & $\frac{L}{c} \to \infty$.

So, it becomes a case of only 1 element which is vertical & having load at pin joint node 1. Hence, as there is only 1 element, so stiffness considered is of 1 element only which can be seen from $u_{y_1} = \frac{-P}{\left(\frac{EA}{L}\right)}$.

d) $\bar{u}^{(e)} = T^{(e)} . u^{(e)}$

local — global.

$$T = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \quad \text{(Rotation matrix)}$$

Deformation – $d^e = \bar{u}^{(e)}_{x_2} - \bar{u}^{(e)}_{x_1}$

Force deformation relation – $F^{(e)} = \left(\frac{EA}{L}\right)^e \times d^{(e)}$

element ① -

$$\bar{u}^{(1)} = T^{(1)} \cdot u^{(1)}$$

$$u^{(1)} = \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \end{bmatrix} \quad \therefore \bar{u}^{(1)} = \begin{bmatrix} -s & c & 0 & 0 \\ -c & -s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -s \cdot u_{x_1} + c \cdot u_{y_1} \\ -c \cdot u_{x_1} - s \cdot u_{y_1} \\ 0 \\ 0 \end{bmatrix}$$

global local.

$$d^{(1)} = \bar{u}^{(1)}_{x_2} - \bar{u}^{(1)}_{x_1} = 0 + s \cdot u_{x_1} - c \cdot u_{y_1}$$

$$\therefore d^{(1)} = \frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)}$$

The force is given by - $F^{(1)} = \left(\frac{EA}{L}\right)^{(1)} \times d^{(1)}$

$$\therefore \boxed{F^{(1)} = \frac{H}{2s} + \frac{Pc^2}{(1+2c^3)}} \qquad = \frac{EA}{L} \times c \times \left[\frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)}\right]$$

element ② -

$$\bar{u}^{(2)} = T^{(2)} \cdot u^{(2)}$$

$$u^{(2)} = \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_3} \\ u_{y_3} \end{bmatrix} \quad \therefore \bar{u}^{(2)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u_{y_1} \\ -u_{x_1} \\ 0 \\ 0 \end{bmatrix}$$

global local.

$$d^{(2)} = \bar{u}^{(2)}_{x_2} - \bar{u}^{(2)}_{x_1} \longleftrightarrow \bar{u}^{(2)}_{x_3} - \bar{u}^{(2)}_{x_1} = 0 - (-u_{x_1}).$$

$$\therefore d^{(2)} = \frac{PL}{EA(1+2c^3)} \qquad \therefore F^{(2)} = \left(\frac{EA}{L}\right)^{(2)} \times d^{(2)}$$

$$\therefore \boxed{F^{(2)} = \frac{P}{1+2c^3}}$$

element ③ -

$$\bar{u}^{(3)} = T^{(3)} \cdot u^{(3)}$$

$$u^{(3)} = \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_4} \\ u_{y_4} \end{bmatrix} \quad \therefore \bar{u}^{(3)} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} s \cdot u_{x_1} + c \cdot u_{y_1} \\ -c \cdot u_{x_1} + s \cdot u_{y_1} \\ 0 \\ 0 \end{bmatrix}$$

global local.

$$d^{(3)} = \bar{u}^{(3)}_{x_2} - \bar{u}^{(3)}_{x_1} \longleftrightarrow \bar{u}_{x_4} - \bar{u}_{x_1} = 0 - s \cdot u_{x_1} - c \cdot u_{y_1}$$

$$\therefore d^{(3)} = -\frac{HL}{2EA \cdot cs} + \frac{PLc}{EA(1+2c^3)}$$

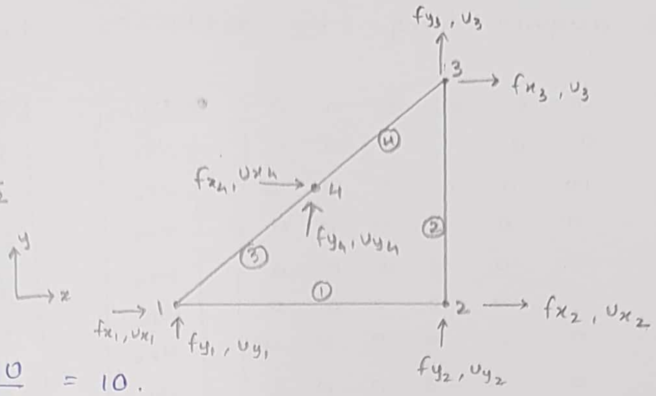
$\therefore F^{(3)} = \left(\frac{EA}{L}\right)^{(3)} \times d^{(3)}$ $\therefore \boxed{F^{(3)} = -\frac{H}{2S} + \frac{Pc^2}{EA(1+2c^3)}}$

If $\alpha \to 0$, $F^{(1)}$ & $F^{(3)}$ both tend to ∞ and give indefinate sol. if $H \neq 0$.

The sol. blows up because the forces tend to infinity, which means that either the structure cannot withstand the forces, or it is not properly restricted.

Assignment 2 -

$E^{(1)}A^{(1)} = 100$, $L^{(1)} = 10$
$E^{(2)}A^{(2)} = 50$, $L^{(2)} = 10$
$E^{(3)}A^{(3)} = 200\sqrt{2}$, $L^{(3)} = 5\sqrt{2}$
$E^{(4)}A^{(4)} = 200\sqrt{2}$, $L^{(4)} = 5\sqrt{2}$



element 1 -

$$\frac{EA}{L} = \frac{100}{10} = 10.$$

$$10\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_{x_1}^{(1)} \\ u_{y_1}^{(1)} \\ u_{x_2}^{(1)} \\ u_{y_2}^{(1)} \end{bmatrix} = \begin{bmatrix} f_{x_1}^{(1)} \\ f_{y_1}^{(1)} \\ f_{x_2}^{(1)} \\ f_{y_2}^{(1)} \end{bmatrix}$$

element 2 -

$$\frac{EA}{L} = \frac{50}{10} = 5$$

$$5\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_{x_2}^{(2)} \\ u_{y_2}^{(2)} \\ u_{x_3}^{(2)} \\ u_{y_3}^{(2)} \end{bmatrix} = \begin{bmatrix} f_{x_2}^{(2)} \\ f_{y_2}^{(2)} \\ f_{x_3}^{(2)} \\ f_{y_3}^{(2)} \end{bmatrix}$$

element 3 -

$$\frac{EA}{L} = \frac{200\sqrt{2}}{5\sqrt{2}} = 40$$

$$40\begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}\begin{bmatrix} u_{x_1}^{(3)} \\ u_{y_1}^{(3)} \\ u_{x_4}^{(3)} \\ u_{y_4}^{(3)} \end{bmatrix} = \begin{bmatrix} f_{x_1}^{(3)} \\ f_{y_1}^{(3)} \\ f_{x_4}^{(3)} \\ f_{y_4}^{(3)} \end{bmatrix}$$

element 4 -

$$\frac{EA}{L} = \frac{200\sqrt{2}}{5\sqrt{2}} = 40$$

$$40\begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}\begin{bmatrix} u_{x_3}^{(4)} \\ u_{y_3}^{(4)} \\ u_{x_4}^{(4)} \\ u_{y_4}^{(4)} \end{bmatrix} = \begin{bmatrix} f_{x_3}^{(4)} \\ f_{y_3}^{(4)} \\ f_{x_4}^{(4)} \\ f_{y_4}^{(4)} \end{bmatrix}$$

Now, after augmenting the matrices, the equ. become-

$$\begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_{x_1} \\ U_{y_1} \\ U_{x_2} \\ U_{y_2} \\ U_{x_3} \\ U_{y_3} \\ U_{x_4} \\ U_{y_4} \end{bmatrix} = \begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_2} \\ f_{y_2} \\ f_{x_3} \\ f_{y_3} \\ f_{x_4} \\ f_{y_4} \end{bmatrix} \quad \text{element 1}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_{x_1} \\ U_{y_1} \\ U_{x_2} \\ U_{y_2} \\ U_{x_3} \\ U_{y_3} \\ U_{x_4} \\ U_{y_4} \end{bmatrix} = \begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_2} \\ f_{y_2} \\ f_{x_3} \\ f_{y_3} \\ f_{x_4} \\ f_{y_4} \end{bmatrix} \quad \text{element 2}$$

$$\begin{bmatrix} 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -20 & -20 & 0 & 0 & 0 & 0 & 20 & 20 \\ -20 & -20 & 0 & 0 & 0 & 0 & 20 & 20 \end{bmatrix} \begin{bmatrix} U_{x_1} \\ U_{y_1} \\ U_{x_2} \\ U_{y_2} \\ U_{x_3} \\ U_{y_3} \\ U_{x_4} \\ U_{y_4} \end{bmatrix} = \begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_2} \\ f_{y_2} \\ f_{x_3} \\ f_{y_3} \\ f_{x_4} \\ f_{y_4} \end{bmatrix} \quad \text{element 3}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & 0 & -20 & -20 & 20 & 20 \\ 0 & 0 & 0 & 0 & -20 & -20 & 20 & 20 \end{bmatrix} \begin{bmatrix} U_{x_1} \\ U_{y_1} \\ U_{x_2} \\ U_{y_2} \\ U_{x_3} \\ U_{y_3} \\ U_{x_4} \\ U_{y_4} \end{bmatrix} = \begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_2} \\ f_{y_2} \\ f_{x_3} \\ f_{y_3} \\ f_{x_4} \\ f_{y_4} \end{bmatrix} \quad \text{element 4}$$

Adding the matrices-

$$\begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ -10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & -5 & 20 & 25 & -20 & -20 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} U_{x_1} \\ U_{y_1} \\ U_{x_2} \\ U_{y_2} \\ U_{x_3} \\ U_{y_3} \\ U_{x_4} \\ U_{y_4} \end{bmatrix} = \begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_2} \\ f_{y_2} \\ f_{x_3} \\ f_{y_3} \\ f_{x_4} \\ f_{y_4} \end{bmatrix}$$

Applying boundary conditions -

$u_{x_1} = u_{y_1} = u_{y_2} = 0.$

$f_{x_2} = 0, \; f_{x_3} = 2, \; f_{y_3} = 1$

The reduced sys. of equ. is -

$$\begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 20 & 20 & -20 & -20 \\ 0 & 20 & 25 & -20 & -20 \\ 0 & -20 & -20 & 40 & 40 \\ 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} u_{x_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The determinant of this master stiffness matrix is zero. Thus, it is singular & it gives an undefined solution. Physically, this means that adding an extra node to the structure makes it unstable because it only adds extra unknowns to the problem giving no new information to solve the problem.

If seen earlier, without adding that extra node, the problem had sufficient conditions to be solved & hence it gave a solution.

Adding extra nodes, just adds up more unknown making it hard to solve & thus making the structure unstable therefore.

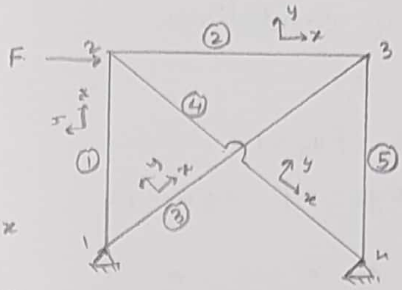
Classwork problem -

$L = 6\,m$

$A = 6\,cm^2 = 6 \times 10^{-4}\,m^2$

$E = 200\,GPa = 2 \times 10^{11}\,Pa$

$F = 80\,kN = 8 \times 10^{4}\,N$



$$K^{(1)} = \frac{2\times10^{11}\times 6\times10^{-4}}{6}\begin{array}{c c}\begin{array}{cccc} ① & & ② & \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} & \begin{array}{c} ① \\ \\ ② \\ \\ \end{array}\end{array} = 10^{7}\begin{array}{c c}\begin{array}{cccc} ① & & ② & \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix} & \begin{array}{c} ① \\ \\ ② \\ \\ \end{array}\end{array}$$

$\angle = 90^\circ$

$$K^{(2)} = 10^{7}\begin{array}{c c}\begin{array}{cccc} ② & & ③ & \end{array} & \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} ② \\ \\ ③ \\ \\ \end{array}\end{array}$$

$\angle = 0^\circ$

$$K^{(5)} = \begin{array}{c c}\begin{array}{cccc} ③ & & ③ & \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix} & \begin{array}{c} ④ \\ \\ ④ \\ \\ \end{array}\end{array} 10^{7}$$

$\angle = 90^\circ$

$$K^{(3)} = 10^{7}\begin{array}{c c}\begin{array}{cccc} ① & & ③ & \end{array} & \\ \begin{bmatrix} 0.7071 & -0.7071 & -0.7071 & 0.7071 \\ -0.7071 & 0.7071 & 0.7071 & -0.7071 \\ -0.7071 & 0.7071 & 0.7071 & -0.7071 \\ 0.7071 & -0.7071 & -0.7071 & 0.7071 \end{bmatrix} & \begin{array}{c} ① \\ \\ ③ \\ \\ \end{array}\end{array}$$

$$K^{(4)} = 10^{7}\begin{array}{c c}\begin{array}{cccc} ② & & ④ & \end{array} & \\ \begin{bmatrix} 0.7071 & 0.7071 & -0.7071 & -0.7071 \\ 0.7071 & 0.7071 & -0.7071 & -0.7071 \\ -0.7071 & -0.7071 & 0.7071 & 0.7071 \\ -0.7071 & -0.7071 & 0.7071 & 0.7071 \end{bmatrix} & \begin{array}{c} ② \\ \\ ④ \\ \\ \end{array}\end{array}$$

After augmenting & adding the matrices, the equ. becomes -

$$10^{7}\begin{bmatrix} 0.7071 & 0.7071 & 0 & 0 & -0.7071 & -0.7071 & 0 & 0 \\ 0.7071 & 2.7071 & 0 & -2 & -0.7071 & -0.7071 & 0 & 0 \\ 0 & 0 & 2.7071 & -0.7071 & -2 & 0 & -0.7071 & 0.7071 \\ 0 & -2 & -0.7071 & 2.7071 & 0 & 0 & 0.7071 & -0.7071 \\ -0.7071 & -0.7071 & -2 & 0 & 2.7071 & 0.7071 & 0 & 0 \\ -0.7071 & -0.7071 & 0 & 0 & 0.7071 & 2.7071 & 0 & -2 \\ 0 & 0 & -0.7071 & 0.7071 & 0 & 0 & 0.7071 & -0.7071 \\ 0 & 0 & 0.7071 & -0.7071 & 0 & -2 & -0.7071 & 2.7071 \end{bmatrix}\begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix} = \begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_2} \\ f_{y_2} \\ f_{x_3} \\ f_{y_3} \\ f_{x_4} \\ f_{y_4} \end{bmatrix}$$

Imposing boundary conditions -

$U_{x_1} = U_{y_1} = U_{x_4} = U_{y_4} = 0.$

$f_{x_2} = 80000$, $f_{y_2} = 0$, $f_{x_3} = 0$, $f_{y_3} = 0$

The reduced sys. of equ. becomes,

$$10^{7}\begin{bmatrix} 2.7071 & -0.7071 & -2 & 0 \\ -0.7071 & 2.7071 & 0 & 0 \\ -2 & 0 & 2.7071 & 0.7071 \\ 0 & 0 & 0.7071 & 2.7071 \end{bmatrix}\begin{bmatrix} U_{x_2} \\ U_{y_2} \\ U_{x_3} \\ U_{y_3} \end{bmatrix} = \begin{bmatrix} 80000 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving equ -

$$\begin{bmatrix} u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \end{bmatrix} = \begin{bmatrix} 8.54 \\ 2.23 \\ 6.77 \\ -1.77 \end{bmatrix} 10^{-3}$$ (all disp. in 'm').