Universitat Politècnica de Catalunya

Computational Structural Mechanics and Dynamics

Master's Degree in Numerical Methods in Engineering

Assignment on the Direct Stiffness Method

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Contents

Abstract

The purpose of this text is to study the behavior of different structures made of elements and nodes. The structures are two-dimensional and under the linear elasticity assumptions. Each joint of the pin-jointed framework will have two degrees of freedom corresponding to the displacements along the two Cartesian axes.

1 Assignment 1

The purpose is to derive the matrix equilibrium equations for the bar elements of the structure. There are three elements to be analyzed. From these, it can be seen in Fig. 1 that elements 1 and 3 are inclined an angle α, whereas element number 2 is disposed vertically. Node 1 is shared by the three elements. In Fig. Fig. 1 the reaction forces have also been included the reactions in point where the displacement is prescribed and the external forces. This structure will be assembled from different components. The elements of the structure and the nodes at which the elements are connected, are numbered, and $\mathbf{u^e} = (u_i, u_j)$ represents the vector of nodal displacements of an element e.

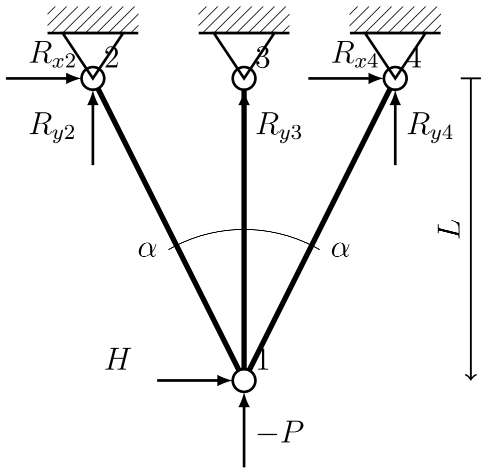

Figure 1: Plane pin-jointed framework of the structure.

The global system of equations will have the following form:

$$\mathbf{Ku} = \mathbf{f} + \mathbf{r} \tag{1}$$

Where,

$$\mathbf{u} = \begin{Bmatrix} \mathbf{u_1} & \mathbf{u_2} & \mathbf{u_3} & \mathbf{u_4} \end{Bmatrix}^T \quad \mathbf{r} = \begin{Bmatrix} H & -P & 0 & 0 & 0 & 0 & 0 & 0 \end{Bmatrix}^T \tag{2}$$

are the displacements of each node and $\mathbf{r}$ is the force of applied nodal forces. For each prescribed displacement a reaction force appears. Then, $\mathbf{f}$ is the vector of nodal forces arising from the distributed loads. In this particular case, there are no distributed body forces.

It is important to note that matrix **K** will always be symmetric.

$$K^{(e)} = \begin{bmatrix} K_{11}^{(e)} & K_{12}^{(e)} \\ K_{21}^{(e)} & K_{22}^{(e)} \end{bmatrix} \tag{3}$$

$$K_{11}^{(e)} = K_{22}^{(e)} = -K_{12}^{(e)} = -K_{21}^{(e)} = (EA/l)^e \begin{bmatrix} cos^2(\beta) & cos(\beta)sin(\beta) \\ cos(\beta)sin(\beta) & sin^2(\beta) \end{bmatrix} \tag{4}$$

where β is the angle formed by each element with the horizontal (taking into account the local numbering, which in this case is in the positive y direction. Therefore $\beta = (\pi/2 + \alpha, \pi/2, \pi/2 - \alpha)$. And taking into account (5), it is possible to construct the stiffness matrices for the elements. Note that each joint contributes a 2×2 matrix.

$$\begin{cases} cos(\pi/2 + \alpha) = -sin(\alpha) \\ cos(\pi/2 - \alpha) = sin(\alpha) \\ sin(\pi/2 + \alpha) = cos(\alpha) \\ sin(\pi/2 - \alpha) = cos(\alpha) \end{cases} \tag{5}$$

It is important to note that for elements 1 and three, the length of the element is $L/cos(\alpha)$. The components of the stiffness matrix for each element will be

$$K_{11}^{(1)} = K_{22}^{(1)} = -K_{12}^{(1)} = -K_{21}^{(1)} = (EA/L) \begin{bmatrix} s^2c & -sc^2 \\ -sc^2 & c^3 \end{bmatrix} \tag{6}$$

$$K_{11}^{(2)} = K_{22}^{(2)} = -K_{12}^{(2)} = -K_{21}^{(2)} = (EA/L) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{7}$$

$$K_{11}^{(3)} = K_{22}^{(3)} = -K_{12}^{(3)} = -K_{21}^{(3)} = (EA/L) \begin{bmatrix} s^2c & sc^2 \\ sc^2 & c^3 \end{bmatrix} \tag{8}$$

With this procedure, it is possible to implement the global stiffness matrix that takes into account the contribution of each element to the global nodes. In doing so, it is obtained

$$\mathbf{K} = \begin{array}{c} \\ 1 \\ 2 \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad\quad 1 \quad\quad & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} \mathrm{K}_{11}^{(1)} + K_{11}^{(2)} + K_{11}^{(3)} & \mathrm{K}_{12}^{(2)} & \mathrm{K}_{12}^{(3)} & 0 \\ & \mathrm{K}_{22}^{(1)} & 0 & 0 \\ & & \mathrm{K}_{22}^{(2)} & 0 \\ \mathit{Symm.} & & & \mathrm{K}_{22}^{(3)} \end{pmatrix} \end{array}$$

Now, if substituting equations (6), (7), (8) into the global stiffness matrix (equation (1), the master stiffness equations are derived straightforwardly. In this system in can clearly be seen that the fifth rows and columns are zeros. This is due to the fact that the structural elements of the problem are bars, which can only handle axial stresses, thus implying that the horizontal reaction on node 3 is zero as the second element is vertically disposed. The x component of the reaction force of the third element corresponds to the fifth row and column of the system, therefore explaining why they contain only zeros.

If removing the rows and columns associated to the elements with prescribed displacements, a 2-equation modified system is obtained and its solution found

$$\begin{cases} \frac{EA}{L}(2cs^2 u_{x1}) = H \longrightarrow u_{x1} = \frac{HL}{2EAcs^2} \\ \frac{EA}{L}(1+2c^3)u_{y1} = -P \longrightarrow u_{y1} = \frac{-PL}{EA(1+2c^3)} \end{cases} \tag{9}$$

When evaluating (9) it is direct to see that if $H \neq 0$ then $u_{x1} \longrightarrow 0$ only if $\alpha \neq 0$. Otherwise this value tends to infinity and it does not make physical sense. For the case in which $\alpha \longrightarrow \pi/2$ both cases make physical sense, as $u_{y1} = -PL/(EA)$ and $u_{x1} \longrightarrow H/(2EA)$ as both c and L tend to zero. If $H = 0$, then there is physical sense for every α.

Now, to analyze the axial forces in the members, Fig. 2 is shown. The force transmitted from element e to nodes $1, 2$ can be calculated as $\mathbf{q^e} = \mathbf{k^e u^e} + \mathbf{f^e}$, where $\mathbf{f^e}$ is zero. Let us start by calculating the given force.

$$\mathbf{u^3} = \begin{Bmatrix} \mathbf{u_1} & \mathbf{u_4} \end{Bmatrix}^T \tag{10}$$

Then,

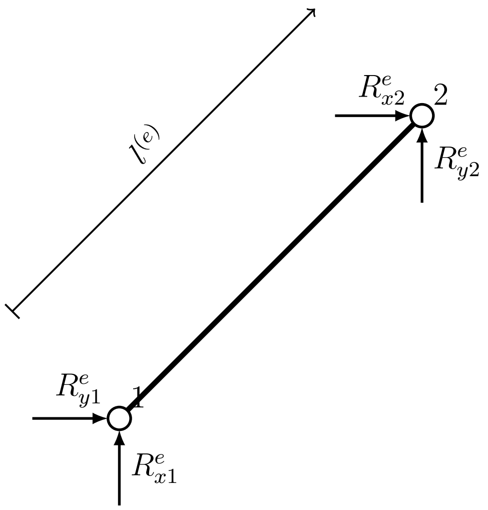

Figure 2: Forces exerted on the nodes of an isolated element.

$$\mathbf{q^3} = \begin{bmatrix} \mathbf{K^{(3)}_{11}} & \mathbf{K^{(3)}_{12}} \\ \mathbf{K^{(3)}_{21}} & \mathbf{K^{(3)}_{22}} \end{bmatrix} \begin{bmatrix} \mathbf{u_1} \\ \mathbf{u_4} \end{bmatrix} = \begin{bmatrix} \frac{H}{2} - \frac{sc^2P}{1+2c^3} \\ \frac{Hc}{2s} - \frac{c^3P}{1+2c^3} \\ \frac{-H}{2} + \frac{sc^2P}{1+2c^3} \\ \frac{-Hc}{2s} + \frac{c^3P}{1+2c^3} \end{bmatrix} \tag{11}$$

$$\mathbf{q^2} = \begin{bmatrix} \mathbf{K^{(2)}_{11}} & \mathbf{K^{(2)}_{12}} \\ \mathbf{K^{(2)}_{21}} & \mathbf{K^{(2)}_{22}} \end{bmatrix} \begin{bmatrix} \mathbf{u_1} \\ \mathbf{u_3} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{P}{1+sc^3} \\ 0 \\ \frac{P}{1+sc^3} \end{bmatrix} \tag{12}$$

$$\mathbf{q^1} = \begin{bmatrix} \mathbf{K^{(1)}_{11}} & \mathbf{K^{(1)}_{12}} \\ \mathbf{K^{(1)}_{21}} & \mathbf{K^{(1)}_{22}} \end{bmatrix} \begin{bmatrix} \mathbf{u_1} \\ \mathbf{u_2} \end{bmatrix} = \begin{bmatrix} \frac{H}{2} + \frac{sc^2P}{1+2c^3} \\ -\frac{Hc}{2s} - \frac{c^3P}{1+2c^3} \\ -\frac{H}{2} - \frac{sc^2P}{1+2c^3} \\ \frac{Hc}{2s} + \frac{c^3P}{1+2c^3} \end{bmatrix} \tag{13}$$

These are the forces that the elements transmit to the nodes expressed in the global system of coordinates, as seen in Fig. 2. To recover the axial forces, it is only needed to project the x or y component of on the axial direction and take into account weather the element is subject to traction or compression. Therefore,

$$\begin{cases} F^1 = \frac{H}{2s} + \frac{c^2 P}{1+2c^3} \longrightarrow Tension \\ F^2 = \frac{P}{1+sc^3} \longrightarrow Tension \\ F^3 = -\frac{H}{2s} + \frac{c^2 P}{1+2c^3} \longrightarrow Compression \quad if \quad \text{H/2s>c}^2 P/(1+2c^3) \end{cases} \tag{14}$$

Again, by observation, it is clear that if $H \neq 0$ then $F_{1,3} \longrightarrow \infty$ if $\alpha \longrightarrow 0$.

Eventually, it can be shown that equilibrium at each node l requires that $\mathbf{r_l} = \sum q_l^e$ along those elements that include node l. At node 1, as it is the only one that has an external force applied, it is clear that:

$$\begin{cases} -\frac{P}{1+sc^3} - \frac{Hc}{2s} - \frac{c^3 P}{1+2c^3} + \frac{Hc}{2s} - \frac{c^3 P}{1+2c^3} = -P \\ \frac{H}{2} + \frac{sc^2 P}{1+2c^3} + \frac{H}{2} - \frac{sc^2 P}{1+2c^3} = H \end{cases} \tag{15}$$

Then, it can also be shown that the whole structure is in equilibrium, as the reactions at the supports balance the action of the external forces.

$$\begin{cases} \frac{sc^2 P}{1+2c^3} - \frac{H}{2} - \frac{H}{2} - \frac{sc^2 P}{1+2c^3} = -H \\ \frac{c^3 P}{1+2c^3} - \frac{Hc}{2s} + \frac{c^3 P}{1+2c^3} + \frac{Hc}{2s} + \frac{P}{1+2c^3} = P \end{cases} \tag{16}$$

2 Assignment 2

The representation of the structure is shown in Fig. 3. Since the numbering of the nodes as well as the material properties are already given, the stiffness matrices of the four elements are the following

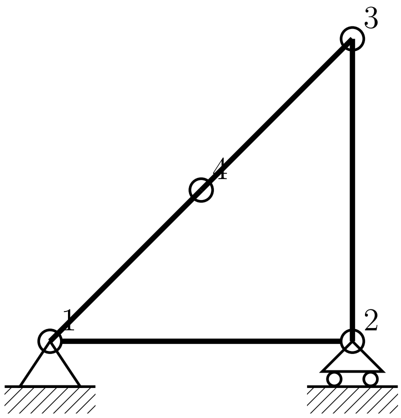

Figure 3: Modified structure with a central node.

$$K^{(1)}_{11} = K^{(1)}_{22} = -K^{(1)}_{12} = -K^{(1)}_{21} = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix} \tag{17}$$

$$K^{(2)}_{11} = K^{(2)}_{22} = -K^{(2)}_{12} = -K^{(2)}_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \tag{18}$$

$$K^{(3)}_{11} = K^{(4)}_{11} = K^{(3)}_{22} = -K^{(3)}_{12} = -K^{(3)}_{21} = \begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix} \tag{19}$$

The master stiffness matrix will be

$$\mathbf{K} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} \mathrm{K}^{(1)}_{11} + K^{(3)}_{11} & \mathrm{K}^{(1)}_{12} & 0 & \mathrm{K}^{(3)}_{12} \\ & \mathrm{K}^{(1)}_{22} + K^{(2)}_{11} & \mathrm{K}^{(2)}_{12} & 0 \\ & & \mathrm{K}^{(4)}_{22} + K^{(2)}_{22} & \mathrm{K}^{(4)}_{21} \\ \mathit{Symm.} & & & \mathrm{K}^{(4)}_{11} + K^{(3)}_{22} \end{pmatrix} \end{array}$$

Now, deleting the 1st, 2nd and 4th rows, it is obtained the simplified master stiffness matrix, which is

$$\mathbf{K} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 10 & -10 & -10 \\ 0 & 10 & 15 & -10 & -10 \\ 0 & -10 & -10 & 20 & 20 \\ 0 & -10 & -10 & 20 & 20 \end{pmatrix} \end{array}$$

This matrix is clearly singular since it has two rows which are combination of one another (actually the fourth and fifth rows are equal) therefore the solution of the system cannot be obtained by inversion of the matrix provided it is singular. This is due to the fact that if a node is placed (with no prescribed displacements) in between an element, the system will present infinite number of positions and therefore it will 'blow up'.

3 Homework

The structural representation is shown in Fig. 4. Again, it is possible to construct the different stiffness matrices for each element using equation (**??**). In doing so, the master stiffness matrix can be constructed.

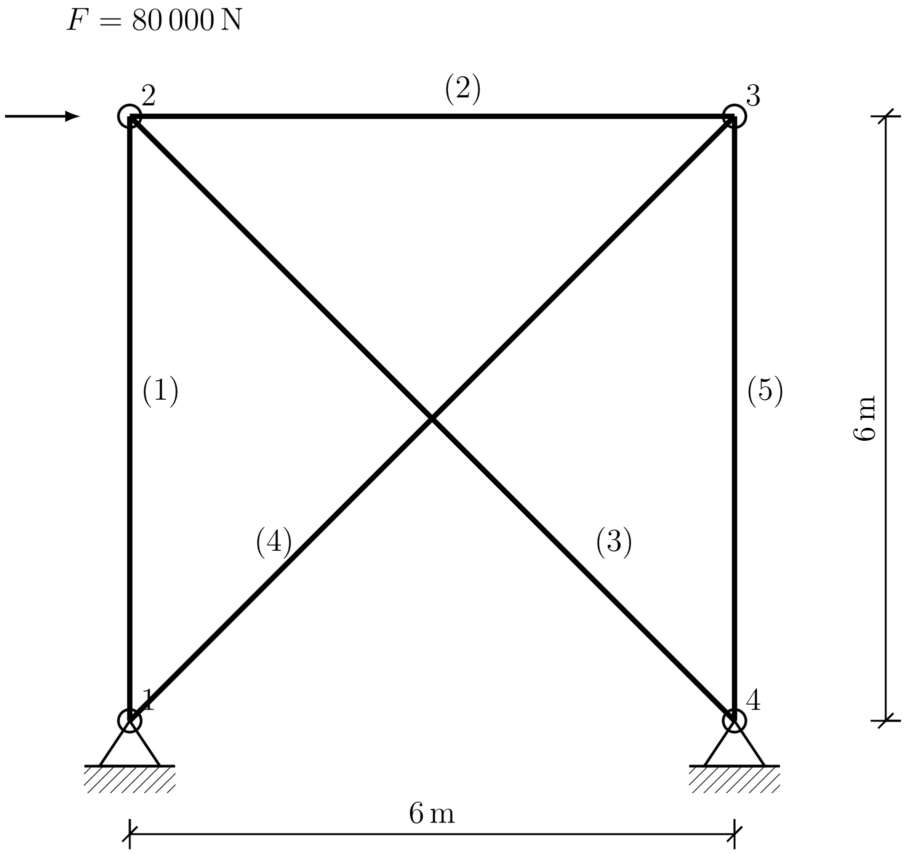

Figure 4: Caption

$$
\mathbf{K} = \begin{array}{c} \\ 1 \\ 2 \\ 2 \\ 4 \end{array}
\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{pmatrix}
\mathrm{K}_{11}^{(1)} + K_{11}^{(4)} & \mathrm{K}_{12}^{(1)} & \mathrm{K}_{12}^{(4)} & 0 \\
\mathrm{K}_{21}^{(1)} & \mathrm{K}_{11}^{(3)} + K_{22}^{(1)} + K_{11}^{(2)} & \mathrm{K}_{12}^{(2)} & \mathrm{K}_{12}^{(3)} \\
\mathrm{K}_{21}^{(4)} & \mathrm{K}_{21}^{(2)} & \mathrm{K}_{22}^{(4)} + K_{22}^{(5)} + K_{22}^{(2)} & \mathrm{K}_{21}^{(5)} \\
0 & \mathrm{K}_{21}^{(3)} & \mathrm{K}_{12}^{(5)} & \mathrm{K}_{22}^{(3)} + K_{11}^{(5)}
\end{pmatrix} \end{array}
$$

For the construction of this matrix, it is important to note that the angles of each element are, from 1 to 5, $(\pi/2, 0, -\pi/4, \pi/4, -\pi/2)$. Then it is important to note that the length of the middle bars will be $6\sqrt{2}m$. With this in mind, it is possible to delete the two first rows and columns as well as the two last in order to consider the prescribed displacement of the supports. With this the following reduced matrix is obtained

$$\mathbf{K}/1\times 10^7 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 2.7071 & -0.7071 & -2 & 0 \\ -0.7071 & 2.7071 & 0 & 0 \\ -2 & 0 & 2.7071 & 0.7071 \\ 0 & 0 & 0.7071 & 2.7071 \end{pmatrix} \end{array}$$

The solution to this system gives the expected solution with the vector force $[F, 0, 0, 0]^T$.