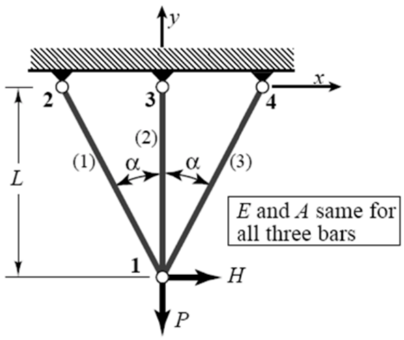
SILVIA PIVETTA (Erasmus)

Assingment 1

Considering the truss problem defined in the figure below, with this geometrical and material properties: $L, \alpha, E \; and \; A, and \; P, H$ are the forces that are applied.



Point 1.

We have 3 trusses and 4 nodes. The structure has 8 degrees of freedom, with six of them removable by the fix-displacement conditions. To analyze this structures we can subdivide the general problem and apply the Direct Stiffness Method and write the element stiffness equation for each of our 3 elements.

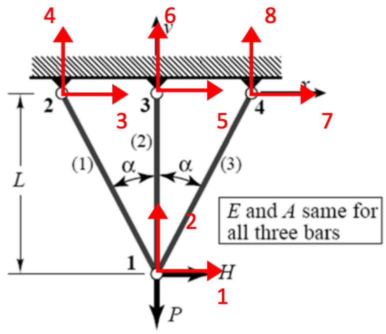
From the theory we know, the element stiffness matrix K^e can be written in terms of the local coordinate element stiffness matrix K^e_{local} and the element rotation matrix T.

$$T^e = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

$$K^e_{local} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^e = (T^e)^T \, K^e_{local} T^e$$

If we apply this last equation we have the element stiffness matrix for each element.



- Element ⟨1⟩ :

$$K_1 = \frac{EA}{L} \cdot c \begin{array}{c} \begin{bmatrix} s^2 & -sc & -s^2 & sc \\ -sc & c^2 & sc & -c^2 \\ -s^2 & sc & s^2 & -sc \\ sc & -c^2 & -sc & c^2 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 7 \\ 8 \end{array}$$

- Element ⟨2⟩ :

$$K_2 = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}$$

- Element ⟨3⟩ :

$$K_3 = \frac{EA}{L} \cdot c \begin{bmatrix} s^2 & sc & -s^2 & -sc \\ sc & c^2 & -sc & -c^2 \\ -s^2 & -sc & s^2 & sc \\ -sc & -c^2 & sc & c^2 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

where $c = cos\alpha$ and $s = sin\alpha$.

Now we can assemble our three element's stiffness matrix to obtain the global stiffness matrix K. To do it we have to apply the compatibility for displacements at joints of the truss and equilibrium between internal and external forces.

$$K_{tot} = K_1 + K_2 + K_3$$

$$\frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ \text{symm} & & & & & & & c^3 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

The master stiffness equation :

$$f = f^1 + f^2 + f^3 = K_{tot}\, u = (K_1 + K_2 + K_3)u$$

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ \text{symm} & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can observe that the 5^{th} row and column contain only zeros. That is because the element 2, which is bar element so it has displacement only along its length, is aligned with the global y-axis, so there is no possible displacement in the global x-axis directions.

Point 2.

By applying the boundary conditions, we can delete the rums and columns where the displacement are zero.

We have

$$u_{x2} = u_{y2} = u_{x3} = u_{y3} = u_{x4} = u_{y4} = 0$$

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ \text{symm} & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ \cancel{u_{x2}} \\ \cancel{u_{y2}} \\ \cancel{u_{x3}} \\ \cancel{u_{y3}} \\ \cancel{u_{x4}} \\ \cancel{u_{y4}} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

The master stiffness equations reduces to this two following equations:

$$H = (2cs^2)u_{x1}\frac{EA}{L}$$
$$-P = (1+2c^3)u_{y1}\frac{EA}{L}$$

Point 3.

We have to solve the two equations in the back to find the displacement of node 1.

$$u_{x1} = \frac{HL}{EA} \cdot \frac{1}{2cs^2}$$

$$u_{y1} = -\frac{PL}{EA} \cdot \frac{1}{1 + 2c^3}$$

We can check how the solution for displacement behave in the limit case:

1- $\alpha \to 0$

$$u_{x1} \to \infty \, ; \, u_{y1} \to -\frac{PL}{3EA}$$

2- $\alpha \to \frac{\pi}{2}$

$$u_{x1} \to \infty \, ; \, u_{y1} \to -\frac{PL}{EA}$$

In the case 1- ($\alpha \to 0$) the displacement u_{x1} tends to infinity because the $sin\alpha$ is at the denominator. This behavior makes physical sense because the elements 1 and 3 tent to come close to the y-axis (we have only one single bar), and this leads to have no stiffness to the force H in the x direction.
As we said before, this behavior is characteristic of the bar element, which has only rigidity along its length direction.
So if $\alpha \to 0$ the three elements overlap and going to coincide with the y axis. In fact the displacement u_{y1} has a finite value that is three times smaller because of the stiffness in the y direction becomes three times bigger.

In the case 2- ($\alpha \to \frac{\pi}{2}$) the displacement u_{x1} tends to infinity because the $cos\alpha$ is at the denominator. The elements 1 and 3 are going to be parallel to the x-axis and their length would tend to infinity, so the stiffness of both elements become zero and there would be no resistance to the force H, so u_{x1} would tend to infinity. Now we have that the only element that is aligned with the y-axis is the element 2, and so this element will have a stiffness and a finite value of the displacement u_{y1}.

Point 4.

We can calculate the displacements according to the local axis of each element by

$$u^e_{local} = T^e u^e$$

and we can calculate the axial forces with the following relationship

$$F^e = \frac{E^e A^e}{L^e} d^e$$

Where $d^e = u_{xj} - u_{xi}$ difference between the axial displacements of the element's node.

- Element ⟨1⟩ :

$$u_{x1} = -\frac{L}{EA}\left(\frac{H}{2sc} + \frac{Pc}{1+2c^3}\right); \quad u_{x2} = 0$$

$$d^1 = u_{x2} - u_{x1} = \frac{L}{EA}\left(\frac{H}{2sc} + \frac{Pc}{1+2c^3}\right);$$

$$F^1 = \left(\frac{H}{2s} + \frac{Pc^2}{1+2c^3}\right)$$

- Element ⟨2⟩ :

$$u_{x1} = -\frac{L}{EA}\left(\frac{P}{1+2c^3}\right); \quad u_{x3} = 0$$

$$d^2 = u_{x3} - u_{x1} = \frac{L}{EA}\left(\frac{P}{1+2c^3}\right);$$

$$F^2 = \left(\frac{P}{1+2c^3}\right)$$

- Element ⟨3⟩ :

$$u_{x1} = \frac{L}{EA}\left(\frac{H}{2sc} - \frac{Pc}{1+2c^3}\right); \quad u_{x4} = 0$$

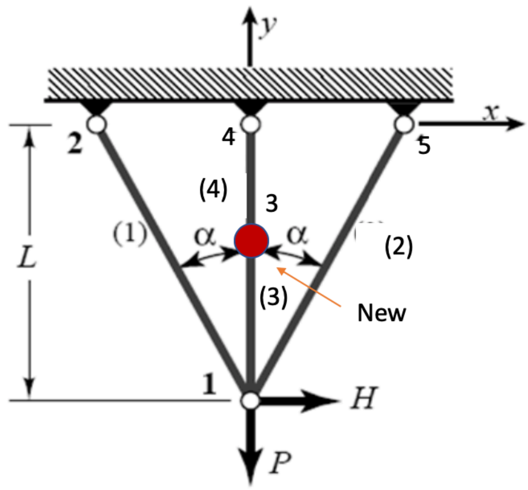
$$d^3 = u_{x4} - u_{x1} = -\frac{L}{EA}\left(\frac{H}{2sc} - \frac{Pc}{1+2c^3}\right);$$

$$F^3 = \left(-\frac{H}{2s} + \frac{Pc^2}{1+2c^3}\right)$$

We must pay attention to the fact that F^3 and F^1 blow up for $\alpha \to 0$ because the horizontal projection of the axial force diminishes as a become smaller. Being H constant, the axial force must increase to compensate.

Assignment 2

Now we have an extra node in the middle of the element 2. So we have another one element.



According with the compatibility for nodal displacements and the equilibrium between internal and external forces, the new master stiffness equations become:

$$\begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & 0 & 0 & -cs^2 & -c^2s \\ 0 & 2+2c^3 & c^2s & -c^3 & 0 & 0 & 0 & -2 & -c^2s & -c^2 \\ -cs^2 & c^2s & cs^2 & -c^2s & 0 & 0 & 0 & 0 & 0 & 0 \\ c^2s & -c^3 & -c^2s & c^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & -2 & 0 & 4 & 0 & 0 \\ -cs^2 & -c^2s & 0 & 0 & 0 & 0 & 0 & 0 & cs^2 & c^2s \\ -c^2s & -c^2 & 0 & 0 & 0 & 0 & 0 & 0 & c^2s & c^2 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \\ u_{x5} \\ u_{y5} \end{bmatrix}$$

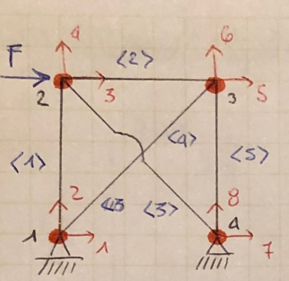
With the boundary conditions:

$$u_{x2} = u_{y2} = u_{x3} = u_{y3} = u_{x4} = u_{y4} = 0$$

$$\begin{bmatrix} H \\ -P \\ 0 \\ 0 \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & 0 & 0 \\ 0 & 2+2c^3 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

This system of equations is singular since the 3rd column is a linear dependent, and it contains only zeros. Row and column are related to the displacement u_{x4}. The bar element 3 is aligned with the global y axis so it could only have displacement in such direction and it also does not have resistance in x direction ; therefore the coefficient in the reduced stiffness matrix related to the displacement of the element 3 in the x direction are zero.

Class exercise 10/02/2020



$L = 6m = 6000\ mm$

$F = 80\ KN = 80000\ N$

$A = 6\ cm^2 = 6000\ mm^2$

$E = 200\ GPa = 200\,000\ \frac{N}{mm^2}$

$EA = 120000000 = 120000\ KN$

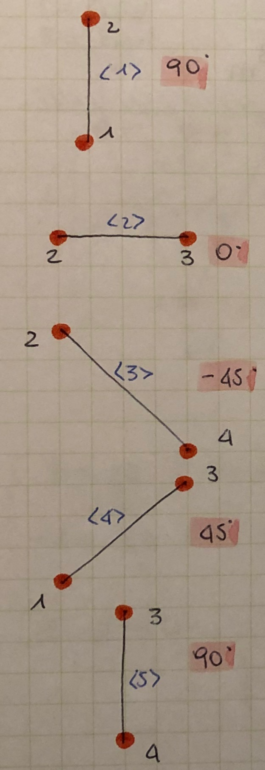
EA/L:

⟨1⟩, ⟨2⟩, ⟨3⟩ = 20000

⟨3⟩, ⟨4⟩ = 14142,13562

$$K^e = \begin{bmatrix} C^2 & SC & -C^2 & -SC \\ & S^2 & -SC & -S^2 \\ & & C^2 & SC \\ & & & S^2 \end{bmatrix}$$

ELEMENT SEPARATION:



$$\begin{array}{c} F \\ \begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2^4 & 0 & -2^4 \\ 0 & 0 & 0 & 0 \\ 0 & -2^4 & 0 & 2^4 \end{bmatrix} \end{array} \begin{array}{c} u \\ \begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$\begin{bmatrix} X_2 \\ Y_2 \\ X_3 \\ Y_3 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 2^4 & 0 & -2^4 & 0 \\ 0 & 0 & 0 & 0 \\ -2^4 & 0 & 2^4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} X_2 \\ Y_2 \\ X_3 \\ Y_3 \end{bmatrix} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$\begin{bmatrix} X_2 \\ Y_2 \\ X_4 \\ Y_4 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} 3 & 4 & 7 & 8 \end{array} \\ \begin{bmatrix} 7071.07 & -7071 & -7071 & 7071 \\ -7071 & 7071 & 7071 & -7071 \\ -7071 & 7071 & 7071 & -7071 \\ 7071 & -7071 & -7071 & 7071 \end{bmatrix} \end{array} \begin{bmatrix} X_2 \\ Y_2 \\ X_4 \\ Y_4 \end{bmatrix} \begin{array}{c} 3 \\ 4 \\ 7 \\ 8 \end{array}$$

$$\begin{bmatrix} X_1 \\ Y_1 \\ X_3 \\ Y_3 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 7071 & 7071 & -7071 & -7071 \\ 7071 & 7071 & -7071 & -7071 \\ -7071 & -7071 & 7071 & 7071 \\ -7071 & -7071 & 7071 & 7071 \end{bmatrix} \end{array} \begin{bmatrix} X_1 \\ Y_1 \\ X_3 \\ Y_3 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}$$

$$\begin{bmatrix} X_3 \\ Y_3 \\ X_4 \\ Y_4 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2^4 & 0 & -2^4 \\ 0 & 0 & 0 & 0 \\ 0 & -2^4 & 0 & 2^4 \end{bmatrix} \end{array} \begin{bmatrix} X_3 \\ Y_3 \\ X_4 \\ Y_4 \end{bmatrix} \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \end{array}$$

GLOBAL SYSTEM

TOT:

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{bmatrix} = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 7071 & 7071 & 0 & 0 & -7071 & -7071 & 0 & 0 \\ 7071 & 27071 & 0 & -20000 & -7071 & -7071 & 0 & 0 \\ 0 & 0 & 27071 & -7071 & -2000 & 0 & -7071 & 7071 \\ 0 & -20000 & -7071 & 27071 & 0 & 0 & 7071 & -7071 \\ -7071 & -7071 & -20000 & 0 & 27071 & 7071 & 0 & 0 \\ -7071 & -7071 & 0 & 0 & 7071 & 27071 & 0 & -20000 \\ 0 & 0 & -7071 & 7071 & 0 & 0 & 7071 & -7071 \\ 0 & 0 & 7071 & -7071 & 0 & -20000 & -7071 & 27071 \end{bmatrix} \end{array} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

BCs: $f_{x1} = f_{x4} = f_{y4} = f_{y1} = \phi$ I detele the rom and column of this members.

I have $f_{x_2} = 80.000$ N
Reduced stiffness due to BCs.

$$\begin{bmatrix} 80.000 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 27071 & -7071 & -20000 & 0 \\ -7071 & 27071 & 0 & 0 \\ -20000 & 0 & 27071 & 7071 \\ 0 & 0 & 7071 & 27071 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

$$\begin{cases} 80000 = 27071\, u_{x2} - 7071\, u_{y2} - 20000\, u_{x3} \\ 0 = -7071\, u_{x2} + 27071\, u_{y2} \\ 0 = -20000\, u_{x2} + 27071\, u_{x3} + 7071\, u_{y3} \\ 0 = 27071\, u_{y3} + 7071\, u_{x3} \end{cases}$$

$$\begin{cases} u_{y3} = -0.2612\, u_{x3} \\ u_{x2} = 1.261\, u_{x3} \\ u_{y2} = 0.3294\, u_{x3} \\ u_{x3} = 6.775 \end{cases} \longrightarrow \text{sol}: \begin{bmatrix} u_{x2} = 8.54 \\ u_{y2} = 2.23 \\ u_{x3} = 6.77 \\ u_{y3} = -1.77 \end{bmatrix} \cdot 10^{-3}\,\text{m}$$